Deformed Toda model coupled to matter: Majorana zero-modes, bound states in-gap and in the continuum.

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Outline

Soliton-fermion system

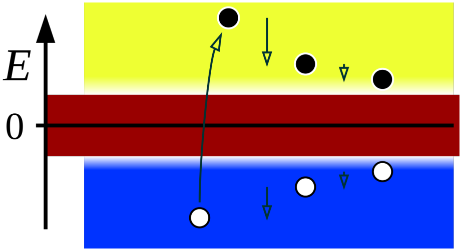
- A solitary wave (a wave packet or pulse) which travels at constant velocity.
- Scalar (soliton) field coupled to spinor (Dirac field).
- Fermion zero-modes trapped to soliton: Charge fractionization of the soliton(half-integer fermion number).
- Fermion bound states and Majorana bound states (zero-modes).
- Bound states in-gap and in the continuum (BIC).

Unifying feature of solitons and trapped fermions

The deformed Toda model coupled to fermion appears in many physical problems, such as particle physics (QCD_2, string theory (D-branes),...), condensed matter physics (superfluidity, superconductivity,...) and cosmological models (cosmic string).

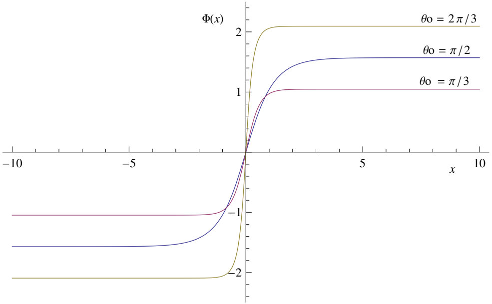
Free Dirac fermion $i\partial\!\!\!/\Psi - M\Psi = 0$,

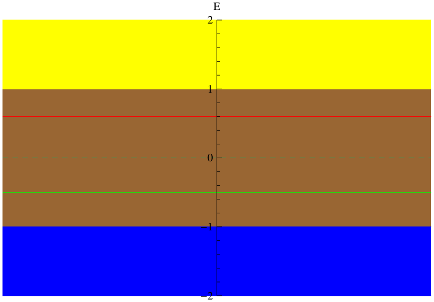
$$\Psi = e^{-iEt}\psi \ \rightarrow \ H_o\psi = E\psi, \ \ H_o = \begin{bmatrix} i\frac{d}{dx} & iM \\ -iM & -i\frac{d}{dx} \end{bmatrix}, \ \ E = \pm\sqrt{k^2 + M^2}$$



Dirac sea for a massive particle. • particles, ∘ antiparticles ('holes' in condensed matter).

Fermion coupled to soliton Φ: $H = \begin{bmatrix} i\frac{d}{dx} & iMe^{-i\beta\Phi(x)} \\ -iMe^{i\beta\Phi(x)} & -i\frac{d}{dx} \end{bmatrix}$, H $\psi = E\psi$





Left: Solitons (kinks) $\Phi(x)$ for $\theta_0 = \pi/2, \theta_0 = \pi/3$, $\theta_0 = 2\pi/3$. Notice that $\Phi(\pm\infty) = \theta_0$.
Right: Spectra of fermion bound states

- **Deformed Toda model coupled to Dirac field**

It has relevant applications in superconductivity, soliton-particle duality

$$\mathcal{L} = \frac{1}{2}\partial_\mu\Phi\,\partial^\mu\Phi + i\bar{\psi}\gamma^\mu\partial_\mu\psi - M\,\bar{\psi}\,e^{i\beta\,\Phi\,\gamma_5}\,\psi - V(\Phi), \tag{1}$$

$$V(\Phi) \equiv A_1\cos(\beta_1\Phi) + A_2\cos(\beta_2\Phi) + A_3\cos(\beta_3\Phi). \tag{2}$$

Notice that $V(\Phi)$ defines a multi-frequency potential
The real parameters β and $\beta_j\,(j = 1, 2, 3)$ are the scalar self-couplings, respectively.
For $V(\Phi) = 0$ (1) defines the integrable Toda model coupled to matter (Dirac) field (TM).
For $V(\Phi) \neq 0$ being a family of potentials (deformed TM), each one defines different physics.

$x_\pm = t \pm x$, and so, $\partial_\pm = \frac{1}{2}(\partial_t \pm \partial_x)$, and $\partial^2 = \partial_t^2 - \partial_x^2 = 4\partial_-\partial_+$. We use $\gamma_0 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$, $\gamma_1 = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}$, $\gamma_5 = \gamma_0\gamma_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, and $\psi = \begin{pmatrix} \psi_R \\ \psi_L \end{pmatrix}$, $\bar{\psi} = \psi^\dagger\gamma_0$.

So, let us consider the Ansatz for $\xi_a (a = 1, 2, 3, 4)$ being real fields

$$\psi_E(x,t) = e^{-iEt}\begin{bmatrix} \xi_3(x) + i\,\xi_4(x) \\ \xi_1(x) + i\,\xi_2(x) \end{bmatrix}. \quad (3)$$

So, the system of equations of motion becomes

$$\begin{aligned}
\xi_1' + E\,\xi_2 - M\xi_4 \sin\beta\Phi + M\xi_3 \cos\beta\Phi &= 0, & (4)\\
\xi_2' - E\,\xi_1 + M\xi_3 \sin\beta\Phi + M\xi_4 \cos\beta\Phi &= 0, & (5)\\
\xi_3' - E\,\xi_4 + M\xi_2 \sin\beta\Phi + M\xi_1 \cos\beta\Phi &= 0, & (6)\\
\xi_4' + E\,\xi_3 - M\xi_1 \sin\beta\Phi + M\xi_2 \cos\beta\Phi &= 0, & (7)\\
\partial_t^2\Phi - \partial_x^2\Phi + 2M\beta\Big[(\xi_1\xi_3 + \xi_2\xi_4)\cos\beta\Phi - (\xi_1\xi_4 - \xi_2\xi_3)\sin\beta\Phi\Big] + V'[\Phi] &= 0, & (8)
\end{aligned}$$

We searh for solutions such that $\Phi(x) = -\Phi(-x)$ with boundary conditions $\Phi(\pm\infty) = \pm\phi_o,\ \ \Phi(0) = 0,\ \ \phi_o = const.,\ \ \xi_a(\pm\infty) \to 0,\ \ a = 1, ..., 4;\ \ \xi_1(0) = \xi_0 = const.$

Majorana zero-modes The Hamiltonian H satisfies $\Gamma^{-1} H \Gamma = H^\star, \quad \Gamma \equiv \pm i\gamma_1$. From $H\xi = E\xi$ and $H^\star \xi^\star = E\xi^\star \Rightarrow \xi = \Gamma\xi^\star$. For Dirac field $\psi_{_E} = e^{-iEt}\xi$ one defines $\psi^\star_{_{-E}} \equiv e^{iEt}\xi^\star$:

$$\psi_{_E} = \Gamma \psi^\star_{_{-E}}. \tag{9}$$

One has charge conjugation symmetry of the system
Particles and holes will have identical spectra.
Particle-hole symmetry (in a relativistic model, a charge conjugation symmetry.)
Majorana fermion is defined for zero-mode solutions ($E = 0$) one can write

$$\psi_{_0} = \Gamma \psi^\star_{_0}. \tag{10}$$

This is a Majorana fermion. These states are well known to lead to interesting phenomena in condensed matter.
The deformed TM above possess these solutions as fermion bound states.

Tau functions (For parameters $\beta_1 = \beta, \;\; \beta_2 = 2\beta, \;\; \beta_3 = 3\beta$)

$$
\begin{aligned}
e^{i\beta\Phi} &= e^{-i\theta_0}\,\frac{\tau_1}{\tau_0}, && (11)\\
\xi_1 &= [\frac{\tau_\xi^1}{\tau_0}]\,e^{-i\theta_1} + [\frac{\widetilde{\tau}_\xi^1}{\tau_1}]\,e^{i\theta_1} && (12)\\
\xi_2 &= -i\{\,[\frac{\tau_\xi^2}{\tau_0}]\,e^{-i\theta_2} - [\frac{\widetilde{\tau}_\xi^2}{\tau_1}]\,e^{i\theta_2}\} && (13)\\
\xi_3 &= [\frac{\tau_\xi^3}{\tau_0}]\,e^{-i\theta_3} + [\frac{\widetilde{\tau}_\xi^3}{\tau_1}]\,e^{i\theta_3} && (14)\\
\xi_4 &= i\{\,[\frac{\tau_\xi^4}{\tau_0}]\,e^{-i\theta_4} - [\frac{\widetilde{\tau}_\xi^4}{\tau_1}]\,e^{i\theta_4}\}, && (15)
\end{aligned}
$$

where $\theta_0,\ \theta_a(a = 1, 2, 3, 4)$ are real parameters.

For bound states consider the tau functions

$$\tau_0 = 1 + e^{-i\theta_0} e^{2\kappa x}, \;\; \tau_1 = 1 + e^{i\theta_0} e^{2\kappa x}, \qquad \kappa, \; \theta_0 \in \mathbb{R} \tag{16}$$

$$\tau_\xi^1 = \rho_1 e^{\kappa x}, \; \tau_\xi^2 = \rho_2 e^{\kappa x}, \; \tau_\xi^3 = \rho_3 e^{\kappa x}, \; \tau_\xi^4 = \rho_4 e^{\kappa x}, \tag{17}$$

$$\widetilde{\tau}_\xi^1 = \rho_1 e^{\kappa x}, \; \widetilde{\tau}_\xi^2 = \rho_2 e^{\kappa x}, \; \widetilde{\tau}_\xi^3 = \rho_3 e^{\kappa x}, \; \widetilde{\tau}_\xi^4 = \rho_4 e^{\kappa x}, \tag{18}$$

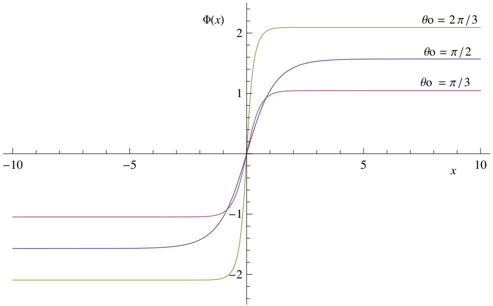
κ and $\theta_0 \in \mathbb{R}$. The fermion components become

$$\xi_a = (-i)^{a-1} \rho_a \, e^{\kappa x - i\theta_a} \Big[\frac{1}{1 + e^{2\kappa x - i\theta_0}} + (-1)^{a-1} \frac{e^{2i\theta_a}}{1 + e^{2\kappa x + i\theta_0}} \Big], \quad a = 1, 2. \tag{19}$$

$\xi_4(x) = -\sigma \xi_1(-x), \; \xi_3(x) = \sigma \xi_2(-x)$. Notice that $\xi_a \in \mathbb{R}$.

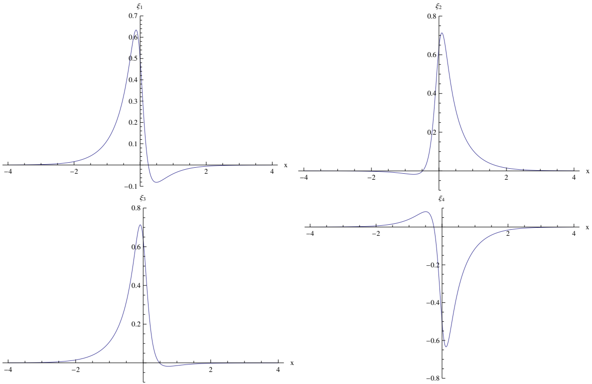
Kinks

$\Phi(x) = \frac{2}{\beta} \arctan \left[\tan\left(\frac{\theta_0}{2}\right) \tanh(\kappa x) \right], \quad \theta_0 \in \mathbb{R}$ The asymptotic values $\Phi(\pm\infty) \equiv \pm\frac{\theta_0}{\beta}$



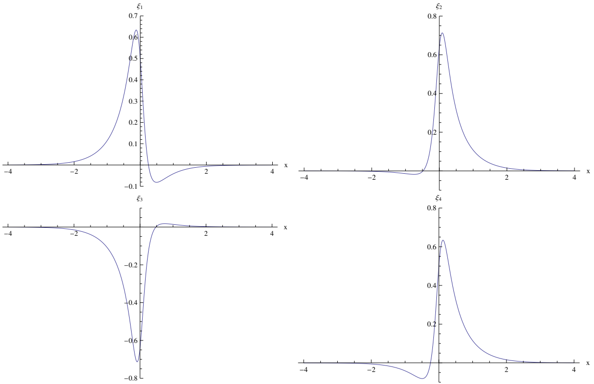
The kinks $\Phi(x)$ for $\{\theta_0 = \pi/2,\ \kappa = 0.656,\ \theta_1 = 0.76,\ \theta_2 = 0.5\}$, $\{\theta_0 = \pi/3,\ \kappa = 1.56,\ \theta_1 = 0.87,\ \theta_2 = 1.7\}$, $\{\theta_0 = 2\pi/3,\ \kappa = 1.96,\ \theta_1 = 0.14,\ \theta_2 = 0.85\}$, $\beta = 1$ and $\xi_0 = 0.5$. Notice that $\Phi(\pm\infty) = \theta_0$.

Positive parity bounds



The bound state components for *positive parity* $\sigma = +1$. Plotted for parameters: $\xi_0,\ \rho_1 = 0.41, \rho_2 = -0.64,\ \theta_1 = 0.14,\ \theta_2 = 0.85,\ \kappa = 1.96, \theta_0 = 2\pi/3$. Notice that $\xi_1(-x) = -\xi_4(x)$ and $\xi_2(-x) = +\xi_3(x)$.

Negative parity bounds



The bound state components for *negative parity* $\sigma = -1$. Plotted for parameters: ξ_0, $\rho_1 = 0.41, \rho_2 = -0.64$, $\theta_1 = 0.14$, $\theta_2 = 0.85$, $\kappa = 1.96, \theta_0 = 2\pi/3$. Notice that $\xi_1(-x) = +\xi_4(x)$ and $\xi_2(-x) = -\xi_3(x)$

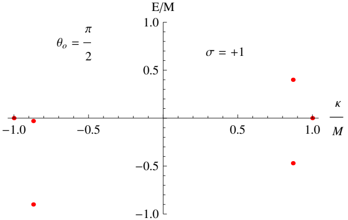
Energies E

Values of E: roots of second order polynomial

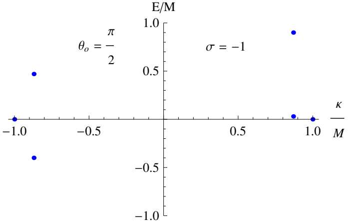
$$[4\kappa^2 - M^2 \sin^2(\theta_o)]E^2 + M\sigma[4\kappa^2 - 2M^2 \sin^2\theta_o - \kappa M\sigma \sin(2\theta_o)]E - M^2 \sin\theta_o[2\kappa M\sigma \cos\theta_o - (\kappa^2 - M^2)\sin\theta_o] = 0. \quad (20)$$

This is a second order polynomial in the variable E. So the exact solutions become $E_{\pm}^{(\sigma)} = -\sigma M + \frac{M}{4}\left\{\frac{2[\sigma \pm \cos\theta_o] + \frac{M}{2\kappa}(1 \pm 2\sigma)\sin\theta_o}{1 - (\frac{M}{2\kappa})^2 \sin^2\theta_0}\right\}$

$\Theta_o=\pi/2,\ \sigma=+1$	
$\frac{E}{M}$	$\frac{\kappa}{M}$
0	±1.0
+0.4	+0.87
-0.03	-0.87
-0.47	+0.87
-0.9	-0.87

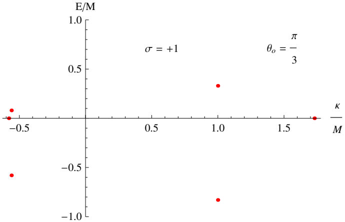


$\Theta_o=\pi/2,\ \sigma=-1$	
$\frac{E}{M}$	$\frac{\kappa}{M}$
0	±1.0
+0.9	+0.87
+0.47	-0.87
+0.03	+0.87
-0.4	-0.87

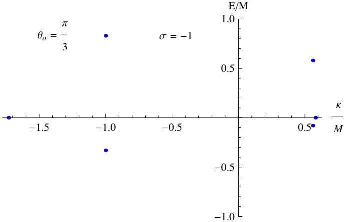


The bound states for $\theta_0 = \pi/2$ and $\sigma = +1$(left) and $\sigma = -1$(right). There is a pair of states ralated by $\sigma \to -\sigma,\ \ \kappa \to -\kappa,\ \ E \to -E$ (a consequence of the particle-hole symmetry of the model). For each parity notice the appearance of two fermion zero-modes(**Majorana** bound states for $E = 0$).

$\Theta_o=\pi/3$, $\sigma=+1$	
$\frac{E}{M}$	$\frac{\kappa}{M}$
0	-0.58
0	+1.73
+0.33	+1
+0.08	-0.56
-0.58	-0.56
-0.83	+1

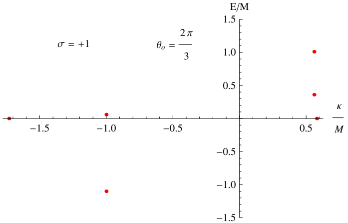


$\Theta_o=\pi/3$, $\sigma=-1$	
$\frac{E}{M}$	$\frac{\kappa}{M}$
0	-1.73
0	+0.58
+0.83	-1
+0.58	+0.56
-0.08	+0.56
-0.33	-1

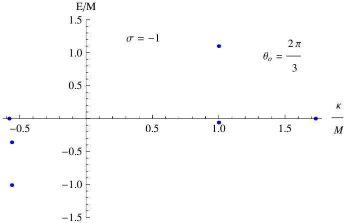


The bound states for $\theta_0 = \pi/3$ and $\sigma = +1$(left) and $\sigma = -1$(right). There is a pair of states ralated by $\sigma \to -\sigma$, $\kappa \to -\kappa$, $E \to -E$ (a consequence of the particle-hole symmetry of the model). For each parity notice the appearance of two fermion zero-modes (**Majorana** bound states for $E = 0$).

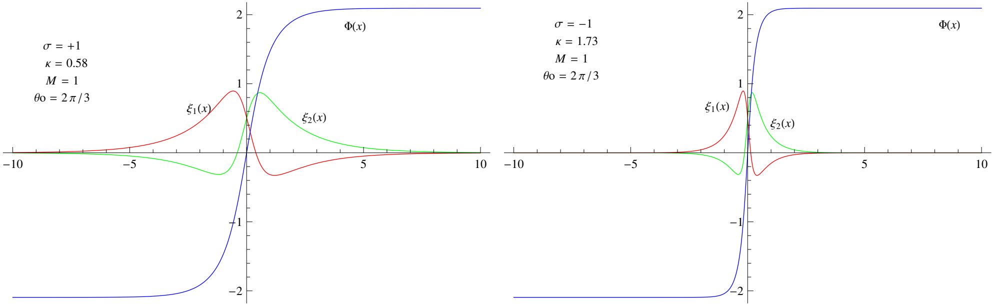
$\Theta_o=2\pi/3$, $\sigma=+1$	
$\frac{E}{M}$	$\frac{\kappa}{M}$
0	-1.73
0	+0.58
+1.01	+0.56
+0.36	+0.56
+0.06	-1
-1.1	-1



$\Theta_o=2\pi/3$, $\sigma=-1$	
$\frac{E}{M}$	$\frac{\kappa}{M}$
0	+1.73
0	-0.58
+1.1	+1
-0.06	+1
-0.36	-0.56
-1.01	-0.56



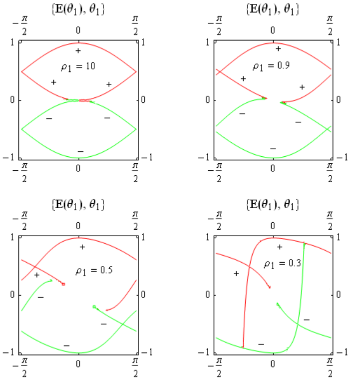
The bound states for $\theta_0 = 2\pi/3$ and parities $\sigma = +1$(left) and $\sigma = -1$(right). There is a pair of states ralated by $\sigma \to -\sigma$, $\kappa \to -\kappa$, $E \to -E$ (a consequence of the particle-hole symmetry of the model). Notice the appearance of the fermion zero-modes (**Majorana** bound states for $E = 0$). Remarkably, for each parity sector there are two states **in the continuum (BIC)** (i.e. states s.t. $E > +M$ or $E < -M$, with $\pm M$ being the threshold states) for energies $E = \pm 1.01$ and $E = \pm 1.1$, respectively.

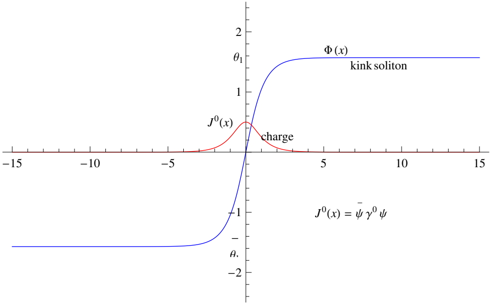


The bound state plots for $E = 0$(zero-modes) and $\theta_0 = 2\pi/3$ and parities $\sigma = +1$(left) and $\sigma = -1$(right).

$\psi_E = \Gamma \psi^{\star}_{-E}$:
Case I ($\Gamma = -i\gamma_1$) $\quad \xi_1 = -\xi_3, \quad \xi_2 = \xi_4,$
Case II ($\Gamma = i\gamma_1$) $\quad \xi_1 = \xi_3, \quad \xi_2 = -\xi_4.$

- Fermion zero modes and topological defects: Drastic change of fermionic vacuum
  Defect itself in some cases acquires fermionic charge and half-integer spin
  Quantum statistics of defect is reversed: becomes a fermion





The bound state energies when $M = 1$.The signs $\pm$ indicate the parity of the bound states. θ_1 is the asymptotic value of the soliton Φ. ρ_1 parameter of the fermion bound state.

- The deformed Toda model coupled to fermion has a rich spectra.
- The spectra comprise: Bound states in-gap (and zero-modes) and in the continuum.
- The appearance of **Majorana fermions** due to particle-hole symmetry.
- The model possesses analytical solutions through the **tau** functions approach.
- The analytical solutions incorporate the back reaction of the fermion on the soliton.
- Potential applications in cosmology and condensed matter physics (quantum computation with majorana fermions).

References

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Thank you !