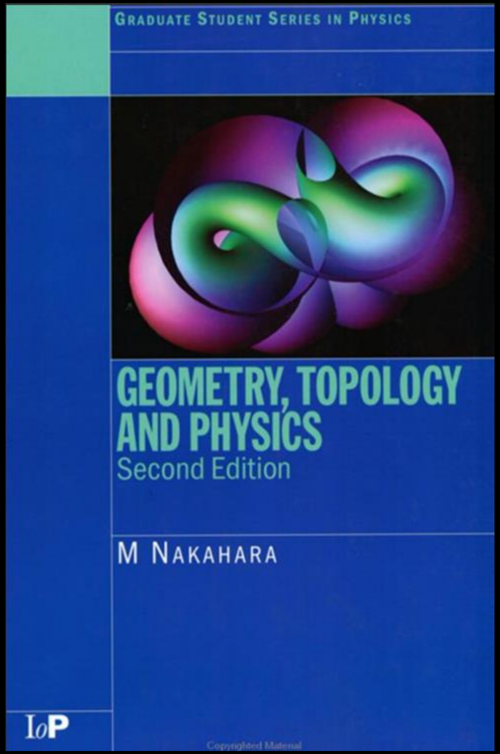
Relatividad General

2024-2

Prof. David Choque Q.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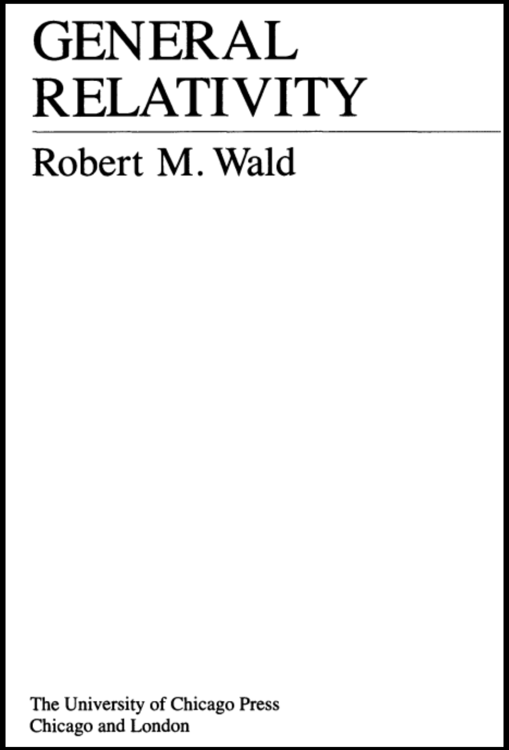
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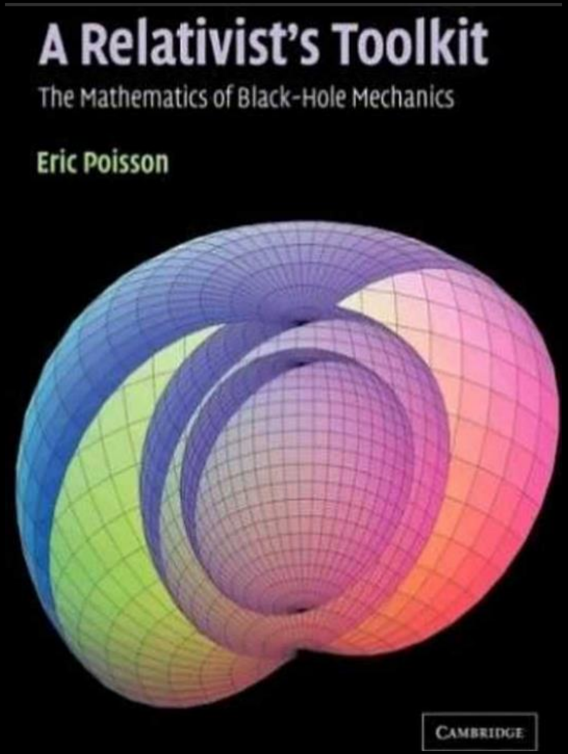
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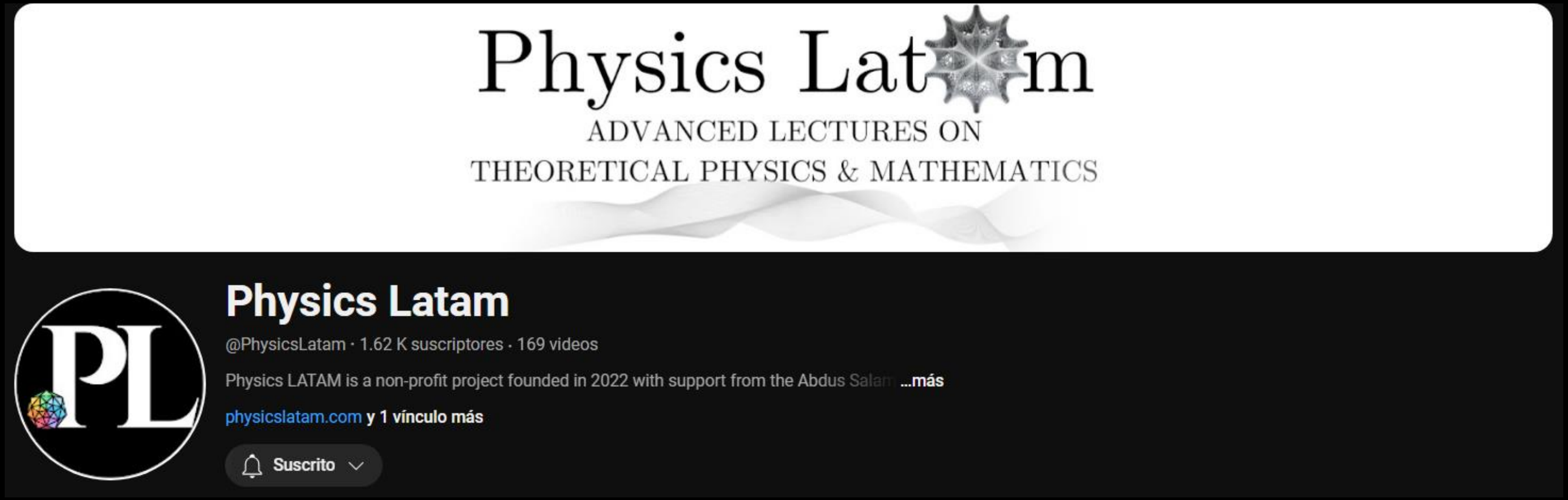
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Suscrito

Base no-coordenada

Base anholonoma

$m = \mathrm{Dim}(M)$

Base coordenada		Base no-coordenada
$T_pM \quad \{\partial_\mu \equiv e_\mu\}$	$\xrightarrow{GL(m;\mathbb{R})}$	$\{e_a\}$
$T^*_pM \quad \{dx^\mu \equiv e^\mu\}$	$e_a{}^\mu$	$\{e^a\}$

$$e_a = e_a{}^{\mu}\partial_\mu$$
$$e^a = e^a{}_\mu dx^\mu$$

$$\langle e^a, e_b\rangle = e^a \cdot e_b = \delta^a{}_b$$

$$g(\partial_\alpha, \partial_\beta) = \partial_\alpha \cdot \partial_\beta = g_{\alpha\beta}$$

$$g(e_a, e_b) = g(e_a{}^{\mu}\partial_\mu, e^{\nu}{}_b \partial_\nu)$$

$$g(e_a, e_b) = e_a{}^{\mu} e^{\nu}{}_b\, g(\partial_\mu, \partial_\nu)$$

$$\eta_{ab} = e^{\mu}{}_a e^{\nu}{}_b\, g_{\mu\nu}$$

$$\eta_{ab} =: (-1 ; +1, \dots +1)$$

$$g(e_a, e_b) = \eta_{ab}$$

$$g(\partial_\mu, \partial_\nu) = g_{\mu\nu}$$

$$e_a{}^\mu e^b{}_\mu = \delta_a{}^b$$

$$e^a{}_\mu e_a{}^\nu = \delta_\mu{}^\nu$$

$$V = V^\mu \partial_\mu = V^a e_a = V^a e_a{}^\mu \partial_\mu$$

$$V^\mu = V^a e_a{}^\mu$$

V^μ: componentes de V en la base coordenada

$$V^a = V^\mu e^a{}_\mu$$

V^a: componentes de V en la base no-coordenada

$$ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu = \eta_{ab}\, e^a{}_\mu\, e^b{}_\nu\, dx^\mu dx^\nu$$

$$ds^2 = \eta_{ab}\, e^a e^b$$

$$e^a{}_\mu =: \left(\quad \right)_{m\times m}$$

$\mathrm{Dim}(\mathcal{M}) = 4 \Rightarrow$ Vierbeins

$\mathrm{Dim}(\mathcal{M}) = m \rightarrow$ Vielbeins

Coeficientes de anholonomia: La nueva base no proviene de una simple transformación de coordenadas locales.

$$[e_a, e_b]\big|_p = C_{ab}{}^c\, e_c\big|_p$$

$$[e_a, e_b] = C_{ab}{}^c\, e_c$$

$$C_{ab}{}^c = [e_a{}^\nu(\partial_\nu e_b{}^\mu) - e_b{}^\nu(\partial_\nu e_a{}^\mu)]\, e^c{}_\mu$$

$$C_{ab}{}^c = -2(\partial_\mu e^c{}_\nu)\, e^\mu{}_{[a}\, e^\nu{}_{b]}$$

Propiedades de los coeficientes de an-holonomia

$$[e_a, e_b] = C_{ab}{}^{c} e_c$$

$$\boxed{C_{ab}{}^{c} \equiv e^{c}{}_{\nu}\left[e_a{}^{\mu}\partial_\mu e_b{}^{\nu} - e_b{}^{\mu}\partial_\mu e_a{}^{\nu}\right]}$$

$$e^{c}{}_{\nu} e_b{}^{\nu} = \delta^{c}{}_{b}$$

$$\partial_\mu\left(e^{c}{}_{\nu} e_b{}^{\nu}\right) = 0 \Rightarrow e^{c}{}_{\nu}\left(\partial_\mu e_b{}^{\nu}\right) = -e_b{}^{\nu}\partial_\mu\left(e^{c}{}_{\nu}\right)$$

$$\boxed{C_{ab}{}^{c} = -2\,\partial_\mu e^{c}{}_{\nu}\, e^{\mu}{}_{[a} e^{\nu}{}_{b]}} = -2\, e^{\mu}{}_{b} e^{\nu}{}_{a}\, \partial_{[\mu} e^{c}{}_{\nu]}$$

$$\boxed{de^{a} = \frac{1}{2}\left(\partial_\mu e^{a}{}_{\nu} - \partial_\nu e^{a}{}_{\mu}\right) dx^{\mu}\wedge dx^{\nu}} \boxed{= -\frac{1}{2} C_{\mu\nu}{}^{a}\, dx^{\mu}\wedge dx^{\nu}}$$

Conexión base coordenada/no-coordenada

$$\nabla_{\partial_\mu}(\partial_\nu) \equiv \nabla_\mu(\partial_\nu) = \Gamma^{\sigma}_{\mu\nu}\,\partial_\sigma$$ → Conexión respecto a la base coordenada $\{\partial_\mu\}$

$$\nabla_{e_a}(e_b) \equiv \nabla_a(e_b) = \Gamma^{c}{}_{ab}\,e_c$$ → Conexión respecto a la base no-coordenada $\{\hat{e}_a\}$

$$\nabla_X Y = \mathrm{Inv} \Longrightarrow \nabla_{e_b} = e^{\mu}_{b}\,\nabla_{\partial_\mu}$$

$X = e_a$

$Y = e_b$

$$\nabla_{e_b}(e_a) = e^{\mu}_{b}\,\nabla_{\partial_\mu}(e_a)$$

$$\nabla_a(e_b) = e^{\sigma}{}_{a}\,\nabla_\sigma\left(e_b{}^{\mu}\,\partial_\mu\right)$$

$$e_b = e^{\mu}_{b}\,\partial_\mu$$

$$\nabla_a = e^{\sigma}{}_{a}\,\nabla_\sigma$$

$$\nabla_a(e_b) = e^{\sigma}{}_{a}\left[\partial_\sigma(e_b{}^{\mu})\,\partial_\mu + e_b{}^{\mu}\,\Gamma^{\lambda}_{\sigma\mu}\,\partial_\lambda\right]$$

$$\nabla_\sigma(e_b{}^{\lambda}) \equiv \partial_\sigma e_b{}^{\lambda} + \Gamma^{\lambda}{}_{\sigma\mu}\,e^{\mu}_{b}$$

$$\nabla_a e_b = e^{\sigma}{}_{a}\,\nabla_\sigma(e_b{}^{\lambda})\,e_\lambda$$

$$\Gamma_{ab}{}^{c}\,e^{\lambda}_{c}\,e_\lambda = e^{\sigma}{}_{a}\,\nabla_\sigma(e^{\lambda}_{b})\,e_\lambda$$

$$\nabla_a(e_b) = \Gamma^{c}{}_{ab}\,e_c$$

$$e^{\lambda}_{c}\,e_\lambda = e_c$$

$$\Gamma_{ab}{}^{c} = e^{c}_{\lambda}\,e^{\sigma}{}_{a}\,\nabla_\sigma(e^{\lambda}_{b})$$

$$\Gamma_{ab}{}^{c} = e^{c}{}_{\lambda}\, e^{\sigma}{}_{a}\, \nabla_\sigma (e^{\lambda}{}_{b})$$

$$\nabla_\sigma (e_b{}^{\lambda}) \equiv \partial_\sigma e_b{}^{\lambda} + \Gamma^{\lambda}{}_{\sigma\mu} e^{\mu}{}_{b}$$

$$\Gamma_{ab}{}^{c} = e^{c}{}_{\lambda}\, e^{\sigma}{}_{a} (\partial_\sigma e^{\lambda}{}_{b} + \Gamma^{\lambda}{}_{\sigma\mu} e^{\mu}{}_{b})$$

Sea la definición

$$\omega_\sigma{}^{c}{}_{b}\, dx^\sigma \equiv \omega^{c}{}_{b}$$

$$\omega^{c}{}_{b} \equiv \Gamma^{c}{}_{ab}\, e^{a}$$

$$\Gamma_{ab}{}^{c} = e^{c}{}_{\lambda}\, e^{\sigma}{}_{a} (\partial_\sigma e^{\lambda}{}_{b} + \Gamma^{\lambda}{}_{\sigma\mu} e^{\mu}{}_{b})$$

$$\Gamma^{c}{}_{ab}\, e^{a} = e^{c}{}_{\lambda}\, dx^\sigma\, \nabla_\sigma (e^{\lambda}{}_{b})$$

$$\omega^{c}{}_{b} = e^{c}{}_{\lambda} (\nabla_\sigma e^{\lambda}{}_{b}) dx^\sigma$$

$$\omega_\sigma{}^{c}{}_{b}\, dx^\sigma = e^{c}{}_{\lambda} (\nabla_\sigma e^{\lambda}{}_{b}) dx^\sigma$$

$$\omega_\sigma{}^{c}{}_{b} = e^{c}{}_{\lambda}\, \nabla_\sigma (e^{\lambda}{}_{b})$$

$$\omega_\sigma{}^{c}{}_{b} = e^{c}{}_{\lambda} (\partial_\sigma e^{\lambda}{}_{b} + \Gamma^{\lambda}{}_{\sigma\mu} e^{\mu}{}_{b})$$

$$e^{b}{}_{\mu}\, e^{\lambda}{}_{b} = \delta^{\lambda}_{\mu}$$

$$-e^{b}{}_{\mu}\, \partial_\sigma e^{\lambda}{}_{b} = e^{\lambda}{}_{b}\, \partial_\sigma e^{b}{}_{\mu}$$

$$\Gamma^{\lambda}{}_{\sigma\mu} = e^{\lambda}{}_{b} (\partial_\sigma e^{b}{}_{\mu} + e^{c}{}_{\mu}\, \omega_\sigma{}^{b}{}_{c})$$

Postulado del vielbein?

Base coordenada

$T_p\mathcal{M}$ $\quad \{\partial_\mu \equiv e_\mu\}$

$T^*_p\mathcal{M}$ $\quad \{dx^\mu \equiv e^\mu\}$

$GL(m;\mathbb{R})$ $\Longrightarrow$

$e_a{}^\mu$

Base no-coordenada

$\{e_a\}$

$\{e^a\}$

Sea la definición $\omega_\sigma{}^c{}_b\, dx^\sigma \equiv \omega^c{}_b$ $\quad \omega^c{}_b = \Gamma^c{}_{ab}\, e^a$

$$\boxed{\Gamma^c{}_{ab} = \Gamma_{ab}{}^c = \omega_\mu{}^c{}_b\, e^\mu_a}$$

$\nabla_X Y = \mathrm{Inv} \Longrightarrow$ $X = e_a$, $Y = e_b$

$\nabla_{e_b} = e^\mu_b \nabla_{\partial_\mu}$

$\nabla_{e_b}(e_a) = e^\mu_b \nabla_{\partial_\mu}(e_a)$

$\nabla_{\partial_\mu}\partial_\nu = \Gamma_{\mu\nu}{}^\sigma \partial_\sigma$ $\qquad \nabla_{\partial_\mu} dx^\nu = -\Gamma^\nu{}_{\mu\sigma}\, dx^\sigma$

$\nabla_{\partial_a} e_b = \Gamma_{ab}{}^c \partial_c$ $\qquad \nabla_{\partial_a} e^b = -\Gamma^b{}_{ac}\, e^c$

$\nabla_{\partial_\mu} e_a = \omega_\mu{}^c{}_a e_c$ $\qquad \nabla_{\partial_\mu} e^a = -\omega_\mu{}^a{}_c\, e^c$

$$\Gamma_{ab}{}^c = e^c{}_\lambda\, e^\sigma_a \left(\partial_\sigma e^\lambda{}_b + \Gamma^\lambda{}_{\sigma\mu}\, e^\mu{}_b\right)$$

$$\omega_\sigma{}^c{}_b = e^c{}_\lambda \left(\partial_\sigma e^\lambda{}_b + \Gamma^\lambda{}_{\sigma\mu}\, e^\mu{}_b\right)$$

$$\Gamma^\lambda{}_{\sigma\mu} = e^\lambda_b \left(\partial_\sigma e^b_\mu + e^c_\mu\, \omega_\sigma{}^b{}_c\right)$$

Postulado del vielveim

Conexión de spin (Lorentz) y derivada covariante

$$\Gamma^c{}_{ab} e^a = \omega^c{}_b = \omega_\mu{}^c{}_b \, dx^\mu$$

$$\Gamma^c{}_{ab} e^a \cdot e_l = \omega^c{}_b \cdot e_l = \omega_\mu{}^c{}_b \, dx^\mu \cdot e_l$$

$$\Gamma^c{}_{ab} = \omega^c{}_b \cdot e_a = \omega_\mu{}^c{}_b \, e^\mu_\rho e^\rho \cdot e_a$$

$$\boxed{\Gamma^c{}_{ab} = \Gamma_{ab}{}^c = \omega_\mu{}^c{}_b \, e^\mu_a}$$

$$\nabla_{e_a}(e_b) = \Gamma_{ab}{}^c e_c$$

$$e^\mu_a \nabla_{\partial_\mu}(e_b) = \Gamma_{ab}{}^c e_c$$

$$\nabla_{\partial_\mu} e_b = \Gamma^c{}_{ab} \, e_c e^a_\mu$$

$$\nabla_{\partial_\mu} e_b = [\omega_\nu{}^c{}_b \, e^\nu_a] e_c e^a_\mu$$

$$\nabla_{\partial_\mu} e_b = \omega_\nu{}^c{}_b \, \delta^\nu_\mu e_c$$

$$\boxed{\nabla_{\partial_\mu} e_b = \Gamma_\mu{}^c{}_b e_c}$$

$$\nabla_{e_a} e^b = -\Gamma^b{}_{ac} e^c$$

$$e^\mu_a \nabla_{\partial_\mu} e^b = -\Gamma^b{}_{ac} e^c$$

$$\nabla_{\partial_\mu} e^b = -\Gamma^b{}_{ac} e^c e^a_\mu$$

$$\nabla_{\partial_\mu} e^b = -\omega_\nu{}^b{}_c \, e^\nu_a e^c e^a_\mu$$

$$\boxed{\nabla_{\partial_\mu} e^b = -\omega_\mu{}^b{}_c \, e^c}$$

$$¿\, D V^b = dV^b + \omega^b{}_a V^a \, ?$$

$$DT^{a_1\ldots a_r}{}_{b_1\ldots b_s} = \mathrm{d}T^{a_1\ldots a_r}{}_{b_1\ldots b_s} + \omega^{a_1}{}_c T^{ca_2\ldots a_r}{}_{b_1\ldots b_s} + \ldots + \omega^{a_r}{}_c T^{a_1\ldots a_{r-1}c}{}_{b_1\ldots b_s} - \omega^c{}_{b_1} T^{a_1\ldots a_r}{}_{cb_2\ldots b_s} - \ldots - \omega^c{}_{b_s} T^{a_1\ldots a_r}{}_{b_1\ldots b_{s-1}c}.$$

$$\nabla_X V = X^\mu \nabla_{\partial_\mu}(V^a e_a)$$

$$\nabla_X V = dx^\mu \nabla_{\partial_\mu}(V^a e_a)$$

$$\nabla_X V = dX^\mu \left[(\partial_\mu V^a) e_a + V^a (\nabla_{\partial_\mu} e_a) \right]$$

$$\nabla_X V = dx^\mu \left[(\partial_\mu V^a) e_a + V^a \omega_\mu{}^b{}_a e_b \right]$$

$$\nabla_X V = dx^\mu \left[\partial_\mu V^b + \omega_\mu{}^b{}_a V^a \right] e_b$$

$$\nabla_X V = (dV^b + \omega^b{}_a V^a) e_b$$

$$\nabla \equiv dx^\mu \nabla_\mu$$

$$\nabla_X V = dx^\mu (\nabla_\mu V^b) e_b$$

$$\nabla_X V = \nabla V^b e_b = D V^b e_b \Rightarrow \nabla V^b \equiv D V^b = dV^b + \omega^b{}_a V^a$$

$$\nabla_X V = dx^\mu \left[\partial_\mu V^b + \omega_\mu{}^b{}_a V^a \right] e_b \,; \quad V = V^a e_a$$

$$V^a \equiv e^a$$

$$\nabla_X e = dx^\mu \left[(\partial_\mu e^b) + \omega_\mu{}^b{}_a e^a \right] e_b$$

$$\nabla_X e = \left[de^b + \omega_\mu{}^b{}_a dx^\mu \wedge e^a{}_\nu dx^\nu \right] e_b$$

$$\nabla_X e = \left[de^b + \omega^b{}_a \wedge e^a \right] e_b$$

$$\nabla_X e = (De^b)\, e_b$$

$$De^b \equiv de^b + \omega^b{}_a \wedge e^a$$

$$\nabla_X K = dx^\mu \nabla_{\partial_\mu} (K^a{}_b e_a \wedge e^b) = dx^\mu \left[(\partial_\mu K^a{}_b) e_a \wedge e^b + K^a{}_b (\nabla_{\partial_\mu} e_a) \wedge e^b + K^a{}_b e_a \wedge (\nabla_{\partial_\mu} e^b) \right]$$

$$\nabla_X K = dx^\mu \left[(\partial_\mu K^a{}_b) e_a \wedge e^b + K^a{}_b \omega_\mu{}^c{}_a e_c \wedge e^b + K^a{}_b e_a \wedge (-\omega_\mu{}^b{}_c e^c) \right]$$

$$\nabla_X K = dx^\mu \left[(\partial_\mu K^a{}_b) e_a \wedge e^b + K^c{}_b \omega_\mu{}^a{}_c e_a \wedge e^b - K^a{}_c \omega_\mu{}^c{}_b e_a \wedge e^b \right]$$

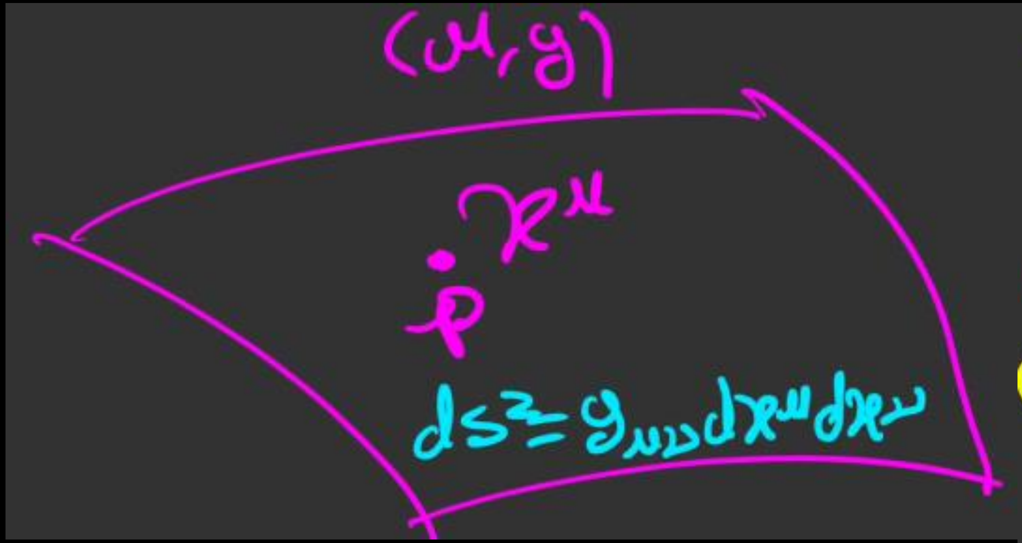
$$\nabla_X K = dx^\mu \left[(\partial_\mu K^a{}_b) + K^c{}_b \omega_\mu{}^a{}_c - K^a{}_c \omega_\mu{}^c{}_b \right] e_a \wedge e^b$$

$$\nabla_X K = (\partial_\mu K_\sigma{}^a{}_b) dx^\mu \wedge dx^\sigma + K^c{}_{\sigma b} \omega_\mu{}^a{}_c dx^\mu \wedge dx^\sigma - K_\sigma{}^a{}_c \omega_\mu{}^c{}_b dx^\mu \wedge dx^\sigma$$

$$\nabla_X K = \left[dK^a{}_b + \omega^a{}_c \wedge K^c{}_b - \omega^c{}_b \wedge K^a{}_c \right] e_a \wedge e^b$$

$$\nabla_X K = DK^a{}_b \, e_a \wedge e^b$$

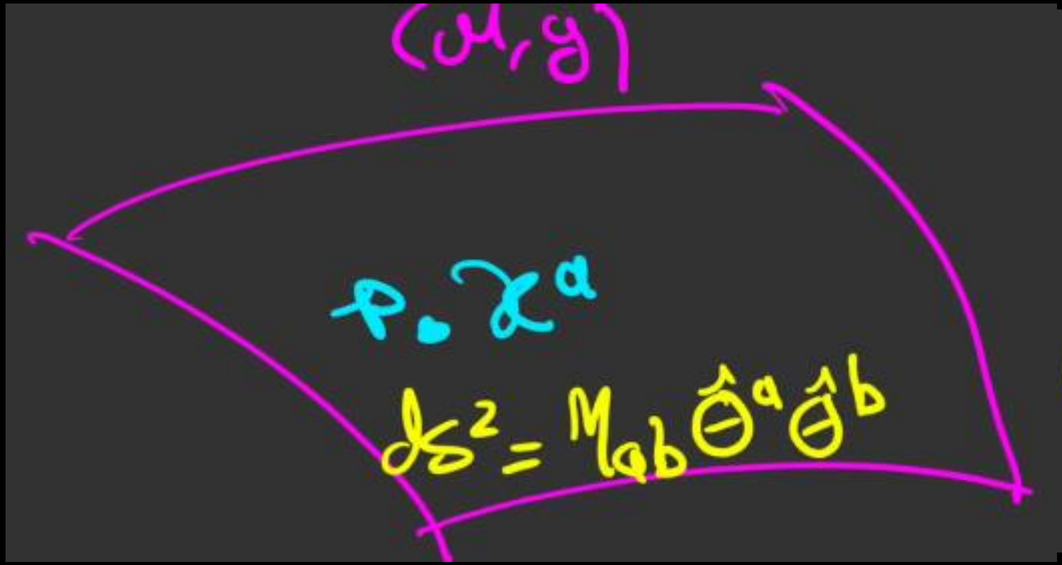
Rotaciones ortogonales locales



$\text{Dim}(\mathcal{M}) = m$

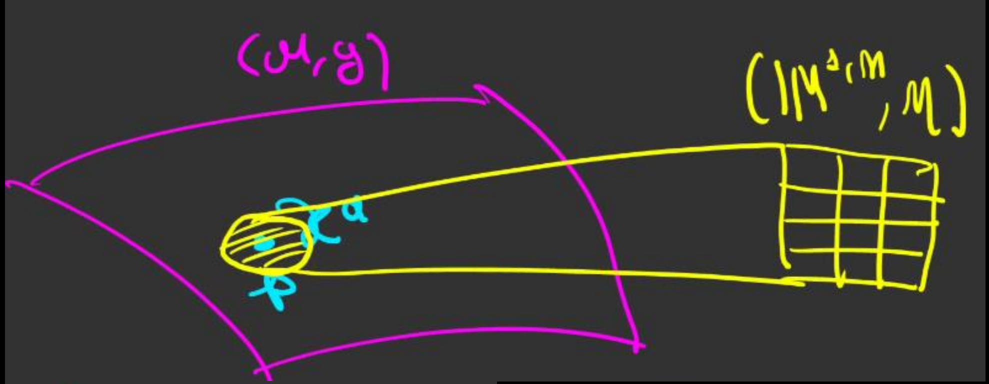
Base coordenada $T_p\mathcal{M} := \{\partial_\mu\}$; $T^*_p\mathcal{M} =: \{dx^\mu\}$

Grados de Libertad $(g_{\mu\nu}) = \dfrac{m(m+1)}{2}$



Base no coordenada : $T_p\mathcal{M} = \{\hat{e}_a\}$; $T^*_p\mathcal{M} := \{\hat{\theta}^a\}$

Grados de Libertad $(e_a{}^\mu) = m^2$?



$$\hat{\theta}^a \Rightarrow \hat{\theta}'^a(p) = \Lambda^a{}_b(p)\,\hat{\theta}^b(p)$$

$$\hat{\theta}'^a = \Lambda^a{}_b\,\hat{\theta}^b(p)$$

$$\hat{\theta}'^a = \Lambda^a{}_b\,\hat{\theta}^b$$

$$ds^2 = \eta_{ab}\,\hat{\theta}^a\,\hat{\theta}^b = \eta'_{cd}\,\hat{\theta}'^c\,\hat{\theta}'^d$$

$$\eta_{ab}\,\hat{\theta}^a\,\hat{\theta}^b = \eta'_{cd}\,\Lambda^c{}_a\,\hat{\theta}^a\,\Lambda^d{}_b\,\hat{\theta}^b$$

$$\eta_{ab}\,\hat{\theta}^a\,\hat{\theta}^b = \eta'_{cd}\,\Lambda^c{}_d\,\Lambda^d{}_b\,\hat{\theta}^a\,\hat{\theta}^b$$

$$\delta_{ab} = \Lambda^c{}_d\,\Lambda^d{}_b\,\delta'_{cd} \Rightarrow \mathcal{M} : \text{Riemann}$$

$$\Lambda^a{}_b(p) \in SO(n)$$

$$\eta_{ab} = \Lambda^c{}_d\,\Lambda^d{}_b\,\eta'_{cd} \Rightarrow \mathcal{M} : \text{Lorentziano}$$

$$\Lambda^a{}_b(p) \in SO(1, n-1)$$

Dimensión de los Grupos de Lie $\begin{matrix} SO(n) \\ SO(1,n-1) \end{matrix} \Rightarrow \dfrac{n(n-1)}{2}$

$$\mathrm{Dim}(g_{\mu\nu}) = \mathrm{Dim}(e^a{}_\mu) - \mathrm{Dim}[SO]$$

$$\frac{n(n+1)}{2} = n^2 - \frac{n(n-1)}{2}$$

Torsion y Curvatura

Torsión en la base no-coordenada

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]$$

$$X = X^a \hat{e}_a \; ; \; Y = Y^b \hat{e}_b$$

$$\nabla_a \hat{e}_b \equiv \nabla_{\hat{e}_a}\hat{e}_b \equiv \Gamma^c{}_{ab}\,\hat{e}_c$$

$$X^a Y^b\, T(\hat{e}_a,\hat{e}_b) = Y^b \nabla_{\hat{e}_a}(\hat{e}_b) - X^a \nabla_{\hat{e}_b}(\hat{e}_a) - X^a Y^b [\hat{e}_a, \hat{e}_b]$$

$$= X^a Y^b \nabla_a(\hat{e}_b) - X^a Y^b \nabla_b(\hat{e}_a) - X^a Y^b C_{ab}{}^c\, \hat{e}_c$$

$$T(\hat{e}_a,\hat{e}_b) = \Gamma_{ab}{}^c\, \hat{e}_c - \Gamma_{ba}{}^c\, \hat{e}_c - C_{ab}{}^c\, \hat{e}_c$$

$$T(\hat{e}_a,\hat{e}_b) = (T_{ab}{}^c)\, \hat{e}_c$$

$$\boxed{T_{ab}{}^c = \Gamma_{ab}{}^c - \Gamma_{ba}{}^c - C_{ab}{}^c} \Longleftrightarrow \boxed{\nabla_a e_b - \nabla_b e_a = [e_a, e_b] + T^c{}_{ab}\, e_c}$$

Riemann en la base no-coordenada

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

$$R(\hat{e}_a,\hat{e}_b)\,\hat{e}_c = R^l{}_{abc}\, \hat{e}_l$$

$$R^l{}_{abc} = \partial_a \Gamma^l{}_{bc} - \partial_b \Gamma^l{}_{ac} + \Gamma^d{}_{bc}\Gamma^l{}_{bc} - \Gamma^d{}_{ac}\Gamma^l{}_{bd} - C_{ab}{}^d\, \Gamma^l{}_{dc}$$

Ecuaciones de Estructura de Cartan

$$de^a + \omega^a{}_b \wedge e^b = T^a$$

$$d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b = R^a{}_b$$

Conexion 1-forma: $\omega^a{}_b \equiv \Gamma^a{}_{cb} e^c$

Torsión 2-forma: $T^a \equiv \frac{1}{2} T^a{}_{bc} e^b \wedge e^c$

Curvatura 2-forma: $R^a{}_b \equiv \frac{1}{2} R^a{}_{bcd} e^c \wedge e^d$

Identidades de Bianchi (En la base no coordenada)

$$d(\ldots) \Rightarrow dT^a + \omega^a{}_b \wedge T^b = R^a{}_b \wedge \theta^b$$

$$d(\ldots) \Rightarrow dR^a{}_b + \omega^a{}_c \wedge R^c{}_b - R^a{}_c \wedge \omega^c{}_b = 0$$

1ra Ecuación de Cartan: jaajaj, es solo la definicion del tensor de torsion

$$\boxed{de^a + \omega^a{}_b \wedge e^b = T^a}$$

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y] \Rightarrow \boxed{T(\partial_\mu,\partial_\nu) = (\Gamma^\sigma{}_{\mu\nu} - \Gamma^\sigma{}_{\nu\mu})\partial_\sigma}$$

$$\boxed{T(\partial_\mu,\partial_\nu) = T^\sigma{}_{\mu\nu}\partial_\sigma}$$

Sea el postulado del vielveim (para no pelear con las Viejas construmbres)

$$\Gamma^\mu{}_{\nu\sigma} = e^\mu{}_b \partial_\nu e^b{}_\sigma + e^\mu{}_b e^c{}_\sigma \omega_\nu{}^b{}_c$$

$$T(\partial_\mu,\partial_\nu) = [e^\sigma{}_b \partial_\mu e^b{}_\nu + e^\sigma{}_b e^c{}_\nu \omega_\mu{}^b{}_c - e^\sigma{}_b \partial_\nu e^b{}_\mu - e^\sigma{}_b e^c{}_\mu \omega_\nu{}^b{}_c]\partial_\sigma$$

$$T_{\mu\nu}{}^\sigma \partial_\sigma = [\partial_\mu e^b{}_\nu - \partial_\nu e^b{}_\mu + e^c{}_\nu \omega_\mu{}^b{}_c - e^c{}_\mu \omega_\nu{}^b{}_c] e^\sigma{}_b \partial_\sigma$$

$$T_{\mu\nu}{}^\sigma e^b{}_\sigma = \partial_\mu e^b{}_\nu - \partial_\nu e^b{}_\mu + e^c{}_\nu \omega_\mu{}^b{}_c - e^c{}_\mu \omega_\nu{}^b{}_c$$

$$T_{\mu\nu}{}^b dx^\mu \wedge dx^\nu = [\partial_\mu e^b{}_\nu - \partial_\nu e^b{}_\mu + e^c{}_\nu \omega_\mu{}^b{}_c - e^c{}_\mu \omega_\nu{}^b{}_c] dx^\mu \wedge dx^\nu$$

$$T_{\mu\nu}{}^b dx^\mu \wedge dx^\nu = 2(\partial_\mu e^b{}_\nu) dx^\mu \wedge dx^\nu + \omega_\mu{}^b{}_c dx^\mu \wedge e^c{}_\nu dx^\nu - e^c{}_\mu dx^\mu \wedge \omega_\nu{}^b{}_c dx^\nu$$

$$T_{\mu\nu}{}^b dx^\mu \wedge dx^\nu = 2[de^b + \omega^b{}_c \wedge e^c]$$

$$\frac{1}{2} T_{\mu\nu}{}^\sigma dx^\mu \wedge dx^\nu = T^b \Rightarrow T^b = de^b + \omega^b{}_c \wedge e^c = De^b$$

Ecuaciones de estructura???

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]$$

$$R(X,Y,Z) = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

Base coordenada

T_pM $\{\partial_\mu \equiv e_\mu\}$

T^*_pM $\{dx^\mu \equiv e^\mu\}$

$$GL(m;\mathbb{R}) \longrightarrow e_a{}^\mu$$

Base no-coordenada

$\{e_a\}$

$\{e^a\}$

$$\boxed{T(\partial_\mu,\partial_\nu) = T^\sigma{}_{\mu\nu}\,\partial_\sigma}$$

$$\boxed{R(\partial_\lambda,\partial_\mu)\partial_\nu = R^\eta{}_{\lambda\mu\nu}\,\partial_\eta}$$

$$R^\eta{}_{\lambda\mu\nu} = \partial\Gamma - \partial\Gamma + \Gamma\Gamma - \Gamma\Gamma$$

$$T^\sigma{}_{\mu\nu} = \Gamma^\sigma{}_{\mu\nu} - \Gamma^\sigma{}_{\nu\mu}$$

$$\Longleftrightarrow$$

$$\boxed{T_{ab}{}^c = \Gamma_{ab}{}^c - \Gamma_{ba}{}^c - C_{ab}{}^c}$$

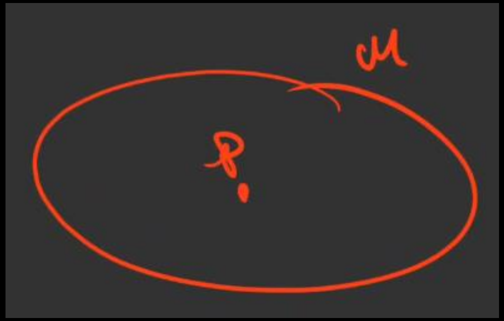
$$T^b = de^b + \omega^b{}_c \wedge e^c$$

$$R^l{}_{abc} = \partial_a \Gamma^l{}_{bc} - \partial_b \Gamma^l{}_{ac} + \Gamma^d{}_{bc}\Gamma^l{}_{bc} - \Gamma^d{}_{ac}\Gamma^l{}_{bd} - C_{ab}{}^d\Gamma^l{}_{dc}$$

$$R^a{}_b = d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b$$

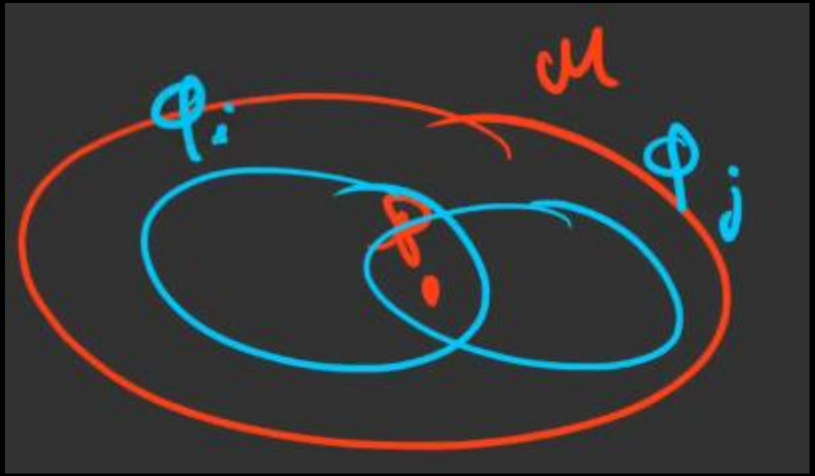
Integración de formas diferenciales

Una integración de una forma diferencial sobre una variedad M se define solo cuando M es 'orientable'



$\varphi_i(p) = \{x^\mu\}$; base de $T_p\mathcal{M}$: $\{\frac{\partial}{\partial x^\mu}\}$

$\varphi_j(p) = \{y^\mu\}$; base de $T_p\mathcal{M}$: $\{\frac{\partial}{\partial y^\mu}\}$



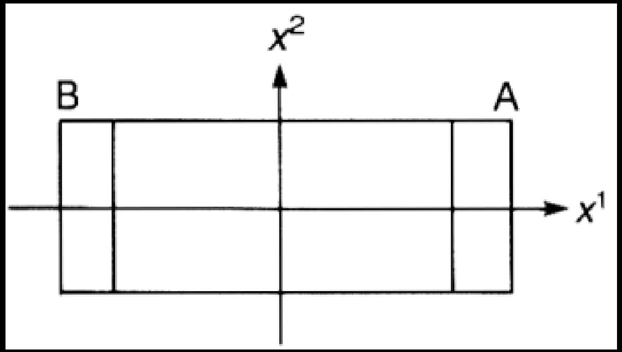
*) $\varphi_i \cap \varphi_j \neq \phi$

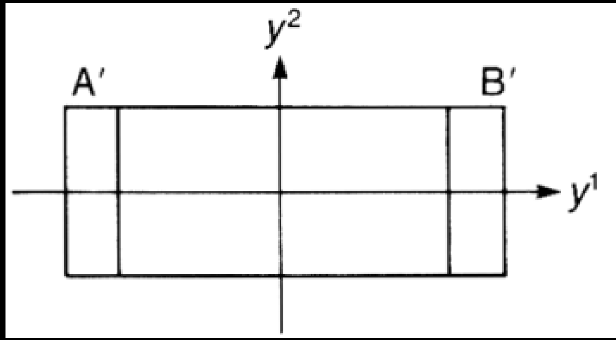
*) $p \in \varphi_i \cap \varphi_j$

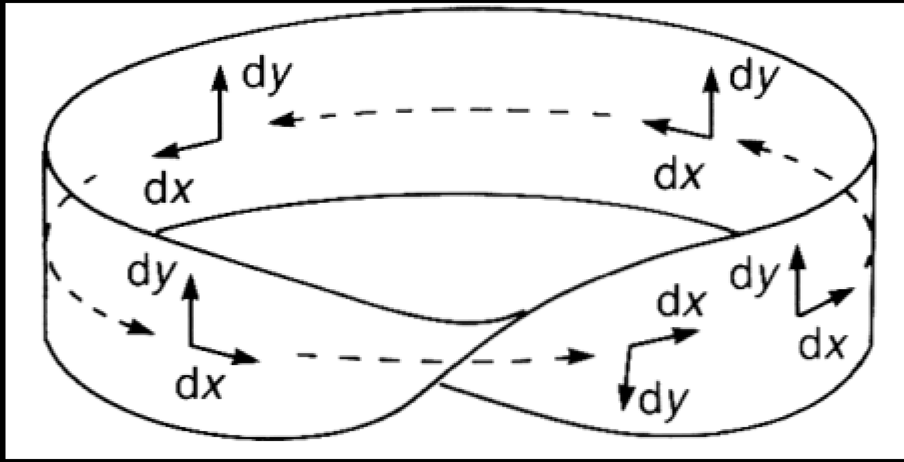
*) $\frac{\partial}{\partial y^\alpha} = \left(\frac{\partial x^\mu}{\partial y^\alpha}\right)\frac{\partial}{\partial x^\mu}$: transformación de la base

Si $\det\left(\frac{\partial x^{\mu}}{\partial y^{\alpha}}\right) > 0$ en $U_i \cap U_j \Rightarrow$ Se dice que las bases $\left\{\frac{\partial}{\partial x^{\mu}}\right\}$ y $\left\{\frac{\partial}{\partial y^{\mu}}\right\}$ definen la misma orientación en $U_i \cap U_j$

Si $\det\left(\frac{\partial x^{\mu}}{\partial y^{\alpha}}\right) < 0$ en $U_i \cap U_j \Rightarrow$ orientación opuesta







$\varphi_1 : \{ y^1 ; y^2 \}$

$\varphi_2 : \{ x^1 ; x^2 \}$

Difeomorfismo:

$y^1(x^1, x^2)$

$y^2(x^1, x^2)$

$$dy^1 = \frac{\partial y^1}{\partial x^1} dx^1 + \frac{\partial y^1}{\partial x^2} dx^2$$

$$dy^2 = \frac{\partial y^2}{\partial x^1} dx^1 + \frac{\partial y^2}{\partial x^2} dx^2$$

$$\begin{pmatrix} dy^1 \\ dy^2 \end{pmatrix} = \begin{pmatrix} \frac{\partial y^1}{\partial x^1} & \frac{\partial y^1}{\partial x^2} \\ \frac{\partial y^2}{\partial x^1} & \frac{\partial y^2}{\partial x^2} \end{pmatrix} \begin{pmatrix} dx^1 \\ dx^2 \end{pmatrix}$$

$$dy^1 \wedge dy^2 = \left(\frac{\partial y^1}{\partial x^1} dx^1 + \frac{\partial y^1}{\partial x^2} dx^2 \right) \wedge \left(\frac{\partial y^2}{\partial x^1} dx^1 + \frac{\partial y^2}{\partial x^2} dx^2 \right)$$

$$dy^1 \wedge dy^2 = \left[\frac{\partial y^1}{\partial x^1} \frac{\partial y^2}{\partial x^2} - \frac{\partial y^1}{\partial x^2} \frac{\partial y^2}{\partial x^1} \right] dx^1 \wedge dx^2$$

$$dy^1 \wedge dy^2 = \det \left(\frac{\partial y^j}{\partial x^\nu} \right) dx^1 \wedge dx^2$$

$$dy^1 \wedge dy^2 \wedge \dots dy^m = \det \left(\frac{\partial y^j}{\partial x^\nu} \right) dx^1 \wedge dx^2 \wedge \dots \wedge dx^m$$

$$dy^1 \wedge dy^2 \wedge \dots dy^m = \det\left(\frac{\partial y^\lambda}{\partial x^\nu}\right) dx^1 \wedge dx^2 \wedge \dots \wedge dx^m$$

Definimos el elemento de volumen: $\Omega_M \equiv \sqrt{-g(x)}\, dx^1 \wedge dx^2 \wedge \dots \wedge dx^m$

El elemento de volumen debe ser invariante, veamos: $\Omega'_M = \Omega_M$

$$\Omega'_M \equiv \sqrt{-g(y)}\, dy^1 \wedge dy^2 \wedge \dots \wedge dy^m$$

$$dy^1 = \frac{\partial y^1}{\partial x^{\mu_1}} dx^{\mu_1}$$

$$dy^2 = \frac{\partial y^2}{\partial x^{\mu_2}} dx^{\mu_2}$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\Omega'_M \equiv \sqrt{-g(y)}\, \frac{\partial y^1}{\partial x^{\mu_1}} dx^{\mu_1} \wedge \frac{\partial y^2}{\partial x^{\mu_2}} dx^{\mu_2} \wedge \dots$$

$$\Omega'_M \equiv \sqrt{-g(y)}\, \frac{\partial y^1}{\partial x^{\mu_1}} \frac{\partial y^2}{\partial x^{\mu_2}} \dots dx^{\mu_1} \wedge dx^{\mu_2} \wedge \dots dx^{\mu_m}$$

$$\Omega'_M \equiv \sqrt{-g(y)}\, \det\left(\frac{\partial y^\lambda}{\partial x^\mu}\right) dx^1 \wedge dx^2 \wedge \dots dx^m$$

Sea la transformación de las componentes del tensor métrico : $g_{\mu\nu}(y) = \frac{\partial x^\lambda}{\partial y^\mu} \frac{\partial x^\sigma}{\partial y^\nu} g_{\lambda\sigma}(x)$

Sea la transformación de las componentes del tensor métrico : $g_{\mu\nu}(y) = \frac{\partial x^\lambda}{\partial y^\mu}\frac{\partial x^\sigma}{\partial y^\nu} g_{\lambda\sigma}(x)$

$$g(y) = \det\left(\frac{\partial x^\mu}{\partial y^\lambda}\right)^2 g(x) \Rightarrow \sqrt{-g(y)} = \pm \det\left(\frac{\partial x^\mu}{\partial y^\lambda}\right)\sqrt{-g(x)}$$

Continuemos con la demostración: $\Omega'_M = \Omega_M$ $\quad \Omega'_M \equiv \sqrt{-g(y)}\,\det\left(\frac{\partial y^\lambda}{\partial x^\mu}\right) dx^1 \wedge dx^2 \wedge \cdots dx^m$

$$\Omega'_M = \pm \cancel{\det\left(\frac{\partial x^\mu}{\partial y^\lambda}\right)} \sqrt{-g(x)}\, \cancel{\det\left(\frac{\partial y^\lambda}{\partial x^\mu}\right)} dx^1 \wedge dx^2 \wedge \cdots \wedge dx^m$$

$$\Omega'_M = \pm \sqrt{-g(x)}\, dx^1 \wedge dx^2 \wedge \ldots \wedge dx^m$$

$$\Omega'_M = \pm \Omega_M \Longrightarrow \Omega'_M = \Omega_M \Leftarrow s + (\text{orientable}) \quad \det\left(\frac{\partial x^\mu}{\partial y^\alpha}\right) > 0$$

El element de volume invariante, si y solo si la variedad es orientable $\quad \Omega_M \equiv \sqrt{-g(x)}\, dx^1 \wedge dx^2 \wedge \ldots \wedge dx^m$

Determinante del tensor métrico y del vielveim:

Dato:

$\{dx^\mu\}$ Base Holonómica

$\{\partial_\mu\}$ $[\partial_\mu, \partial_\nu] = 0$

$\{e^a\}$ Base Anholonómica

$\{e_a\}$ $[e_a, e_b] = C^c{}_{ab} e_c$

Sea la base no-coordenada:

$$\{\theta^a\} = \{e^a{}_\mu dx^\mu\}$$

$$g_{\mu\nu} = \eta_{ab} e^a{}_\mu e^b{}_\nu \Rightarrow \det(g_{\mu\nu}) = -1 \det(e^a{}_\mu)^2$$

$$g = -e^2$$

$$\sqrt{-g} = e$$

$$g \equiv \det(g_{\mu\nu})$$

$$e \equiv \det(e^a{}_\mu)$$

$$e^{-1} = \det(e_a{}^\mu)$$

$$e = \sqrt{-g}$$

$$d\Omega_M = e \, dx^1 \wedge dx^2 \wedge \ldots \, dx^m$$

$$dx^\mu = e_a{}^\mu e^a$$

$$dx^1 = e^1{}_a e^a$$

$$dx^2 = e^2{}_b e^b$$

$$\vdots \quad \vdots \quad \vdots$$

$$d\Omega_M = e \, e_a{}^1 e^2{}_b \ldots e^1 \wedge e^2 \wedge \ldots e^m$$

$$d\Omega_M = e \, e_a{}^1 e^2{}_b \ldots e^1 \wedge e^2 \wedge \ldots e^m$$

$$d\Omega_M = e^1 \wedge e^2 \wedge \ldots \wedge e^m$$

Levi-Civita

Símbolo de Levi-civita

$$\epsilon_{\mu_1\mu_2\ldots\mu_n} = \begin{cases} +1 & \text{even permutations} \\ -1 & \text{odd permutations} \\ 0 & \text{repetido} \end{cases}$$

Tensor de Levi-Civita

$$\varepsilon_{\alpha\beta} = \sqrt{-g}\,\epsilon_{\alpha\beta}$$

En una carta $\varphi_1 = \{x^\mu\} \Rightarrow \epsilon_{\alpha\beta}(x)$

En una carta $\varphi_2 = \{y^\mu\} \Rightarrow \epsilon_{\alpha\beta}(y)$

$$\epsilon_{\alpha\beta}(x)\frac{\partial x^\alpha}{\partial y^\sigma}\frac{\partial x^\beta}{\partial y^\lambda} = \lambda\cdot\epsilon_{\sigma\lambda}(y)$$

$$\epsilon_{\alpha\beta}\frac{\partial x^\alpha}{\partial y^1}\frac{\partial x^\beta}{\partial y^2} = \lambda\cdot\cancel{\epsilon_{12}}\ 1$$

$$\epsilon_{\alpha\beta}\frac{\partial x^\alpha}{\partial y^1}\frac{\partial x^\beta}{\partial y^2} = \lambda$$

$$\det(M) = \epsilon_{\alpha\beta}M^{1\alpha}M^{2\beta}$$

$$\det\left(\frac{\partial x^\lambda}{\partial y^\sigma}\right) = \lambda = \text{Jacobiano}$$

$$g(y) = \det\left(\frac{\partial x^\lambda}{\partial y^\sigma}\right)^2 g(x)$$

$$\lambda = \left(\frac{g(y)}{g(x)}\right)^{1/2}$$

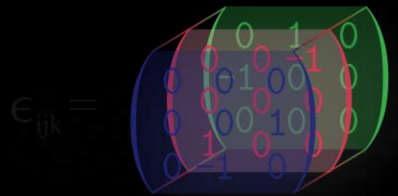
$$\epsilon_{\alpha\beta}(x)\frac{\partial x^\alpha}{\partial y^\sigma}\frac{\partial x^\beta}{\partial y^\lambda} = \lambda\cdot\epsilon_{\sigma\lambda}(y)$$

$$\epsilon_{\alpha\beta}(x)\frac{\partial x^\alpha}{\partial y^\sigma}\frac{\partial x^\beta}{\partial y^\lambda} = \left(\frac{g(y)}{-g(x)}\right)^{1/2}\epsilon_{\sigma\lambda}(y)$$

$$\epsilon_{\alpha\beta}(x)\sqrt{-g(x)}\,\frac{\partial x^\alpha}{\partial y^\sigma}\frac{\partial x^\beta}{\partial y^\lambda} = \epsilon_{\sigma\lambda}(y)\sqrt{-g(y)}$$

$$\varepsilon_{\alpha\beta}\frac{\partial x^\alpha}{\partial y^\sigma}\frac{\partial x^\beta}{\partial y^\lambda} = \epsilon_{\sigma\lambda}(y)\sqrt{-g(y)} = \varepsilon_{\sigma\lambda}$$

D=2



D=m

$$\epsilon_{\alpha\beta}\begin{cases}\epsilon_{12}=+1\\ \epsilon_{21}=-1\\ \epsilon_{11}=\epsilon_{22}=0\end{cases}\quad\Rightarrow\quad \epsilon_{\alpha\beta\cdots}=\begin{cases}\epsilon_{12\cdots\cdots}=+1\\ \epsilon_{21\cdots\cdots}=-1\\ \epsilon_{\cdots\alpha\alpha\cdots}=0\end{cases}$$

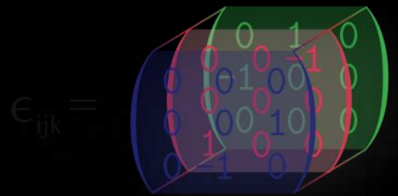
$$\epsilon^{\alpha\beta}\begin{cases}\epsilon^{12}=-1\\ \epsilon^{21}=+1\\ \epsilon^{11}=\epsilon^{22}=0\end{cases}\quad\Rightarrow\quad \epsilon^{\alpha\beta\cdots}=\begin{cases}\epsilon^{12\cdots\cdots}=-1\\ \epsilon^{21\cdots\cdots}=+1\\ \epsilon^{\cdots\alpha\alpha\cdots}=0\end{cases}$$

$$\epsilon_{\alpha\beta\cdots}=-\epsilon^{\alpha\beta\cdots}$$

$$\det(M)=\epsilon_{\alpha\beta}M^{1\alpha}M^{2\beta}\quad\Rightarrow\quad \det(M)=\epsilon_{\mu_1\cdots\mu_m}M^{1\mu_1}M^{2\mu_2}\cdots M^{m\mu_m}$$

$$\det(M)=-\epsilon^{\alpha\beta}M_{1\alpha}M_{2\beta}\quad\Rightarrow\quad \det(M)=-\epsilon^{\mu_1\cdots\mu_m}M_{1\mu_1}M_{2\mu_2}\cdots M_{m\mu_m}$$

$$\begin{aligned}\varepsilon^{\alpha\beta}&=e^{-1}\epsilon^{\alpha\beta}\\ \varepsilon_{\alpha\beta}&=e\,\epsilon_{\alpha\beta}\end{aligned}\quad\Rightarrow\quad\begin{aligned}\varepsilon_{\mu_1\cdots\mu_m}&=\sqrt{-g}\,\epsilon_{\mu_1\cdots\mu_m}\\ \varepsilon^{\mu_1\cdots\mu_m}&=\frac{1}{\sqrt{-g}}\epsilon^{\mu_1\cdots\mu_m}\end{aligned}$$



D=2		D=m
$g_{\alpha\gamma} g_{\beta\delta} \varepsilon^{\alpha\beta} = \varepsilon_{\gamma\delta}$	$\Rightarrow$	$g_{\mu_1\beta_1} g_{\mu_2\beta_2} \cdots g_{\mu_m\beta_m} \varepsilon^{\mu_1\cdots\mu_m} = \varepsilon_{\beta_1\cdots\beta_m}$
$g^{\alpha\gamma} g^{\beta\delta} \varepsilon_{\alpha\beta} = \varepsilon^{\gamma\delta}$	$\Rightarrow$	$g^{\mu_1\beta_1} g^{\mu_2\beta_2} \cdots g^{\mu_m\beta_m} \varepsilon_{\mu_1\cdots\mu_m} = \varepsilon^{\beta_1\cdots\beta_m}$
$\epsilon^{\alpha\beta} = -g\, g^{\alpha\gamma} g^{\beta\delta} \epsilon_{\gamma\delta}$	$\Rightarrow$	$-g\; g^{\mu_1\beta_1} g^{\mu_2\beta_2} \cdots g^{\mu_m\beta_m} \epsilon_{\mu_1\cdots\mu_m} = \epsilon^{\beta_1\cdots\beta_m}$
$\epsilon_{\alpha\beta} = -g^{-1} g_{\alpha\gamma} g_{\beta\delta} \epsilon^{\gamma\delta}$	$\Rightarrow$	$-g^{-1}\, g_{\mu_1\beta_1} g_{\mu_2\beta_2} \cdots g_{\mu_m\beta_m} \epsilon^{\mu_1\cdots\mu_m} = \epsilon_{\beta_1\cdots\beta_m}$

$$\epsilon_{\mu\nu} = \begin{vmatrix} \delta^1_\mu & \delta^1_\nu \\ \delta^2_\mu & \delta^2_\nu \end{vmatrix} = \delta^{1\,2}_{\mu\,\nu}$$

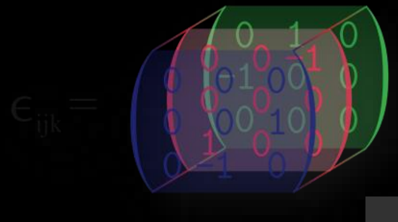
$$= 2!\, \delta^1_{[\mu} \delta^2_{\nu]} = 2!\, \delta^{[1}_{\mu} \delta^{2]}_{\nu}$$

$$\Rightarrow$$

$$\epsilon_{\mu_1\cdots\mu_m} = \begin{vmatrix} \delta^1_{\mu_1} & \cdots & \delta^1_{\mu_m} \\ \vdots & \ddots & \vdots \\ \delta^m_{\mu_1} & \cdots & \delta^m_{\mu_m} \end{vmatrix} = \delta^{1\cdots m}_{\mu_1\cdots\mu_m}$$

$$= m!\, \delta^1_{[\mu_1} \cdots \delta^m_{\mu_m]} = m!\, \delta^{[1}_{\mu_1} \cdots \delta^{m]}_{\mu_m}$$

D=2



D=m

$$\epsilon^{\mu\nu} = -\begin{vmatrix} \delta^{\mu}_{1} & \delta^{\mu}_{2} \\ \delta^{\nu}_{1} & \delta^{\nu}_{2} \end{vmatrix} = -\delta^{\mu\nu}_{12}$$

$$= -2!\,\delta^{\mu}_{[1}\delta^{\nu}_{2]} = -2!\,\delta^{[\mu}_{1}\delta^{\nu]}_{2}$$

$$\epsilon^{\mu_1\cdots\mu_m} = -\begin{vmatrix} \delta^{\mu_1}_{1} & \cdots & \delta^{\mu_1}_{m} \\ \vdots & \ddots & \vdots \\ \delta^{\mu_m}_{1} & \cdots & \delta^{\mu_m}_{m} \end{vmatrix} = -\delta^{\mu_1\cdots\mu_m}_{1\cdots m}$$

$$= -m!\,\delta^{\mu_1}_{[1}\cdots\delta^{\mu_m}_{m]} = -m!\,\delta^{[\mu_1}_{1}\cdots\delta^{\mu_m]}_{m}$$

$$\epsilon_{\mu\nu}\epsilon^{\beta\gamma} = -\begin{vmatrix} \delta^{\beta}_{\mu} & \delta^{\gamma}_{\mu} \\ \delta^{\beta}_{\nu} & \delta^{\gamma}_{\nu} \end{vmatrix} = -\begin{vmatrix} \delta^{\beta}_{\mu} & \delta^{\beta}_{\nu} \\ \delta^{\gamma}_{\mu} & \delta^{\gamma}_{\nu} \end{vmatrix}$$

$$\epsilon_{\mu\nu}\epsilon^{\beta\gamma} = -\delta^{\beta\gamma}_{\mu\nu} = -2!\,\delta^{[\beta}_{\mu}\delta^{\gamma]}_{\nu}$$

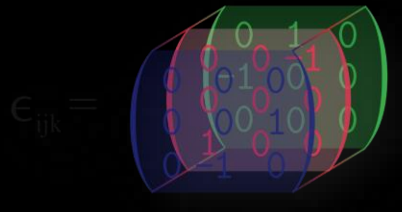
$$= -2!\,\delta^{\beta}_{[\mu}\delta^{\gamma}_{\nu]}$$

$$\epsilon_{\alpha_1\cdots\alpha_m}\epsilon^{\beta_1\cdots\beta_m} = -\begin{vmatrix} \delta^{\beta_1}_{\alpha_1} & \cdots & \delta^{\beta_1}_{\alpha_m} \\ \vdots & \ddots & \vdots \\ \delta^{\beta_m}_{\alpha_1} & \cdots & \delta^{\beta_m}_{\alpha_m} \end{vmatrix} = -\delta^{\beta_1\cdots\beta_m}_{\alpha_1\cdots\alpha_m}$$

$$= -m!\,\delta^{\beta_1}_{[\alpha_1}\cdots\delta^{\beta_m}_{\alpha_m]}$$

$$= -m!\,\delta^{[\beta_1}_{\alpha_1}\cdots\delta^{\beta_m]}_{\alpha_m}$$

D=2



D=m

$$\epsilon_{\mu\nu}\epsilon^{\mu\gamma} = -2!$$

$$\epsilon_{\mu_1\cdots\mu_m\nu_1\cdots\nu_m}\,\epsilon^{\mu_1\cdots\mu_m\nu_1\cdots\nu_m} = -m!$$

D=3

$$\epsilon^{\mu_1\mu_2\nu}\,\epsilon_{\mu_1\mu_2\delta} = -2!\,\delta^{\nu}_{\delta}$$

$$\epsilon^{\mu_1\mu_2\alpha\beta}\,\epsilon_{\mu_1\mu_2\delta\gamma} = -2!\,\delta^{\alpha\beta}_{\delta\gamma}$$

$$\epsilon^{\mu_1\mu_2\mu_3\alpha}\,\epsilon_{\mu_1\mu_2\mu_3\gamma} = -3!\,\delta^{\alpha}_{\gamma}$$

D=p+q

$$\epsilon^{\mu_1\cdots\mu_p\nu_1\cdots\nu_q}\,\epsilon_{\mu_1\cdots\mu_p\sigma_1\cdots\sigma_q} = -p!\,\delta^{\nu_1\cdots\nu_q}_{\sigma_1\cdots\sigma_q}$$

D=2

D=p

$$\delta^{\mu_1\mu_2}_{\nu_1\nu_2}\,\delta^{\nu_1\nu_2}_{\kappa_1\kappa_2} = 2!\,\delta^{\mu_1\mu_2}_{\kappa_1\kappa_2}$$

$$\delta^{\mu_1\cdots\mu_p}_{\nu_1\cdots\nu_p}\,\delta^{\nu_1\cdots\nu_p}_{\kappa_1\cdots\kappa_p} = p!\,\delta^{\mu_1\cdots\mu_p}_{\kappa_1\cdots\kappa_p}$$

$$\delta^{\mu\nu}_{\alpha\beta}\,A^{\alpha\beta} = 2!\,A^{[\mu\nu]}$$

$$\delta^{\mu_1\cdots\mu_p}_{\nu_1\cdots\nu_p}\,A^{\nu_1\cdots\nu_p} = p!\,A^{[\mu_1\cdots\mu_p]}$$

$$\delta^{\mu_1\cdots\mu_p}_{\nu_1\cdots\nu_p}\,A_{\mu_1\cdots\mu_p} = p!\,A_{[\nu_1\cdots\nu_p]}$$

Dual de Hodge y elemento de Volumen

$$* : \Omega^{r}(M) \to \Omega^{m-r}(M)$$

$$*(dx^{\mu_1} \wedge dx^{\mu_2} \wedge \ldots \wedge dx^{\mu_r}) = \frac{\sqrt{-g}}{(m-r)!} \epsilon^{\mu_1 \mu_2 \ldots \mu_r}{}_{\nu_{r+1} \ldots \nu_m} dx^{\nu_{r+1}} \wedge \ldots \wedge dx^{\nu_m}$$

Sea una matriz : $(A^{a}{}_{b})_{m \times m}$

$$(\det A)\, \epsilon_{b_1 \ldots b_m} = \epsilon_{a_1 \ldots a_m} A^{a_1}{}_{b_1} \cdots A^{a_m}{}_{b_m}$$

Sea $(A^{a}{}_{b})_{m \times m} \equiv (e^{a}{}_{\mu})_{m \times m}$

$$\boxed{\begin{aligned} \epsilon_{\mu_1 \ldots \mu_m} &= e^{-1} \epsilon^{a_1}{}_{\mu_1} \cdots \epsilon^{a_m}{}_{\mu_m} \epsilon_{a_1 \ldots a_m} \\ \epsilon^{\mu_1 \ldots \mu_m} &= e\, \epsilon_{a_1}{}^{\mu_1} \cdots \epsilon_{a_m}{}^{\mu_m} \epsilon^{a_1 \ldots a_m} \end{aligned}}$$

$$* =: \frac{1}{(m-r)!} \epsilon^{a_1 \ldots a_r}{}_{b_{r+1} \ldots b_m} e^{b_{r+1}} \wedge \ldots \wedge e^{b_m}$$

Elemento de volumen

$$* : \Omega^{0}(M) \to \Omega^{m-0}(M) \quad ; \quad \operatorname{Dim}(M) = m$$

$$*1 = \frac{\sqrt{-g}}{m!} \epsilon_{\mu_1 \mu_2 \ldots \mu_m} dx^{\mu_1} \wedge dx^{\mu_2} \wedge \ldots \wedge dx^{\mu_m}$$

$$*1 = \sqrt{-g}\; dx^{1} \wedge dx^{2} \wedge \ldots \wedge dx^{m}$$

$$*1 = d\Omega_M$$

Teorema:

Sean: $\omega, \eta \in \Omega^p(\mathcal{M})$ $\qquad p \leq m$

$\omega^*, \eta^* \in \Omega^p(\mathcal{M})$; $m = \text{Dim}(\mathcal{M})$

$$\omega = \frac{1}{p!} \omega_{\mu_1 \cdots \mu_p} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p}$$

$$\eta = \frac{1}{p!} \eta_{\mu_1 \cdots \mu_p} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p}$$

$$\omega \wedge * \eta = \eta \wedge * \omega = \frac{1}{r!} \omega_{\mu_1 \cdots \mu_r} \eta^{\mu_1 \cdots \mu_r} \sqrt{-g}\, d^m x$$

Definicion de producto interior:

Sean: $\omega, \eta \in \Omega^p(\mathcal{M})$ ⇒

$$(\omega, \eta) = \int_{\mathcal{M}} \omega \wedge * \eta = \frac{1}{p!} \int \omega_{\mu_1 \cdots \mu_r} \eta^{\mu_1 \cdots \mu_r} \sqrt{-g}\, d^m x$$

$$(\omega, \eta) = (\eta, \omega)$$

Accion de Hilbert-Einstein

$$S = \frac{1}{2\kappa^2}\int \epsilon_{abcd} R^{cd} \wedge e^a \wedge e^b = \frac{1}{2\kappa^2}\int \epsilon_{abcd}\, e^a \wedge e^b \wedge R^{cd}$$

$$\delta S = \frac{1}{2\kappa^2}\int 2\,\epsilon_{abcd}\,\delta e^a \wedge e^b \wedge R^{cd} + \frac{1}{2\kappa^2}\int \epsilon_{abcd}\, e^a \wedge e^b \wedge \delta R^{cd}$$

Termino de borde

$$\delta S = \frac{1}{2\kappa^2}\int 2\,(\epsilon_{abcd}\, e^b \wedge R^{cd}) \wedge \delta e^a = 0$$

$$\epsilon_{abcd}\, e^b \wedge R^{cd} = 0$$

$$\epsilon_{abcd} e^b \wedge e^f \wedge e^g \left(\frac{1}{2} R^{cd}{}_{fg}\right) = 0$$

$$\epsilon_{abcd} \epsilon^{bfgh} \left(\frac{\Omega(e_h)}{2} R^{cd}{}_{fg}\right) = 0$$

$$-1!\,3!\,\delta^f_{[a}\delta^g_c\delta^h_{d]} \left(\frac{\Omega(e_h)}{2} R^{cd}{}_{fg}\right) = 0$$

$$\left[\delta^f_a\delta^g_c\delta^h_d - \delta^f_a\delta^g_d\delta^h_c + \delta^f_d\delta^g_a\delta^h_c - \delta^f_c\delta^g_a\delta^h_d - \delta^f_d\delta^g_c\delta^h_a + \delta^f_c\delta^g_d\delta^h_a\right]\left(R^{cd}{}_{fg}\right) = 0$$

$$\boxed{4R^h{}_a - 2R\delta^h_a = 0 \rightarrow R^h{}_a - \frac{1}{2}R\delta^h_a = 0}$$

Descomposicion de la conexion

$$de^c + w^c{}_b \wedge e^b = T^c$$
$$dw^a{}_b + w^a{}_c \wedge w^c{}_b = R^a{}_b$$

Conexion de Levi-civita

$$\boxed{\omega^c{}_b = \mathring{\omega}^c{}_b}$$

$$de^c + \dot{w}^c{}_b \wedge e^b = 0$$
$$d\dot{w}^a{}_b + \dot{w}^a{}_c \wedge \dot{w}^c{}_b = \dot{R}^a{}_b$$

$$\Gamma^\mu{}_{\nu\sigma} = \mathring{\Gamma}^\mu{}_{\nu\sigma} + K^\mu{}_{\nu\sigma}$$

$$\Gamma^\mu{}_{\nu\sigma} = e^\mu{}_b \partial_\nu e^b{}_\sigma + e^\mu{}_b e^c{}_\sigma \omega_\nu{}^b{}_c$$

Postulado de Vielveim (ahora sabemos que no es un postulado)

$$\mathring{\Gamma}^\mu{}_{\nu\sigma} = e^\mu{}_b \partial_\nu e^b{}_\sigma + e^\mu{}_b e^c{}_\sigma \mathring{\omega}_\nu{}^b{}_c$$

$$\cancel{e^\mu{}_b \partial_\nu e^b{}_\sigma} + e^\mu{}_b e^c{}_\sigma \omega_\nu{}^b{}_c = \cancel{e^\mu{}_b \partial_\nu e^b{}_\sigma} + e^\mu{}_b e^c{}_\sigma \mathring{\omega}_\nu{}^b{}_c + K^\mu{}_{\nu\sigma}$$

$$\omega_\nu{}^b{}_c = \mathring{\omega}_\nu{}^b{}_c + K^\mu{}_{\nu\sigma} e^b{}_\mu e^\sigma{}_c$$

$$\omega_\nu{}^b{}_c = \mathring{\omega}_\nu{}^b{}_c + K^b{}_{\nu c}$$

$$\boxed{\omega^b{}_c = \mathring{\omega}^b{}_c + K^b{}_c}$$

Descomposicion del tensor de Riemann

$$dw^a{}_b + w^a{}_c \wedge w^c{}_b = R^a{}_b$$

$$w^a{}_b = \overset{\bullet}{w}{}^a{}_b + K^a{}_b$$

$$d(\overset{\bullet}{w}{}^a{}_b + K^a{}_b) + (\overset{\bullet}{w}{}^a{}_c + K^a{}_c) \wedge (\overset{\bullet}{w}{}^c{}_b + K^c{}_b) = R^a{}_b$$

$$\left[d\overset{\bullet}{w}{}^a{}_b + \overset{\bullet}{w}{}^a{}_c \wedge \overset{\bullet}{w}{}^c{}_b\right] + \left[dK^a{}_b + \overset{\bullet}{w}{}^a{}_b \wedge K^c{}_b + K^a{}_c \wedge \overset{\bullet}{w}{}^c{}_b\right] + K^a{}_c \wedge K^c{}_b = R^a{}_b$$

$$\overset{\bullet}{R}{}^a{}_b + \overset{\bullet}{D}K^a{}_b + K^a{}_c \wedge K^c{}_b = R^a{}_b$$

$$\overset{\bullet}{D}K^a{}_b = dK^a{}_b + \overset{\bullet}{\omega}{}^a{}_b \wedge K^c{}_b + K^a{}_c \wedge \overset{\bullet}{\omega}{}^c{}_b$$

$$de^c + w^c{}_b \wedge e^b = T^c$$

$$dw^a{}_b + w^a{}_c \wedge w^c{}_b = R^a{}_b$$

Conexion de Levi-civita

$$de^c + \overset{\bullet}{w}{}^c{}_b \wedge e^b = 0$$

$$d\overset{\bullet}{w}{}^a{}_b + \overset{\bullet}{w}{}^a{}_c \wedge \overset{\bullet}{w}{}^c{}_b = \overset{\bullet}{R}{}^a{}_b$$

$$w^a{}_b = \overset{\bullet}{w}{}^a{}_b + K^a{}_b$$

$$\overset{\bullet}{R}{}^a{}_b + \overset{\bullet}{D}K^a{}_b + K^a{}_c \wedge K^c{}_b = R^a{}_b$$

$$\mathcal{T}^a = \mathcal{K}^a{}_b \wedge e^b$$

Función de Lagrange en una teoría de Einstein-Cartán

$$\mathcal{L} = \frac{1}{4\kappa^2}\epsilon_{abcd}\, e^a \wedge e^b \wedge \overset{\bullet}{R}{}^{cd}$$

$$\overset{\bullet}{R}{}^a{}_b + \overset{\bullet}{D}K^a{}_b + K^a{}_c \wedge K^c{}_b = R^a{}_b$$

$$\mathcal{L} = \frac{1}{4\kappa^2}\epsilon_{abcd}\, e^a \wedge e^b \wedge \left(R^{cd} - \overset{\bullet}{D}K^{cd} - K^c{}_e \wedge K^{ed}\right)$$

Einstein-Cartán

$$\mathcal{L} = \frac{1}{4\kappa^2}\epsilon_{abcd}\, e^a \wedge e^b \wedge \left(R^{cd} - \dot{D}K^{cd} - K^c{}_e \wedge K^{ed}\right)$$

$$w^a{}_b \equiv 0$$

$$\Gamma^\mu{}_{\nu\sigma} = e^\mu_b \partial_\nu e^b_\sigma$$

Conexion de Weitzenböck

$$de^c + w^c{}_b \wedge e^b = T^c$$

$$dw^a{}_b + w^a{}_c \wedge w^c{}_b = R^a{}_b$$

$$\nabla g = 0 \Rightarrow w^a{}_b = \dot{w}^a{}_b + K^a{}_b$$

$$\dot{R}^a{}_b + \dot{D}K^a{}_b + K^a{}_c \wedge K^c{}_b = R^a{}_b$$

$$\mathcal{T}^a = \mathcal{K}^a{}_b \wedge \mathrm{e}^b$$

$$de^c = T^c \qquad R^c{}_d = 0 \Rightarrow K^c{}_e = -\dot{w}^c_e$$

$$\dot{R}^a{}_b + \dot{D}K^a{}_b + K^a{}_c \wedge K^c{}_b = 0$$

$$\mathcal{L} = \frac{1}{4\kappa^2}\epsilon_{abcd}\, e^a \wedge e^b \wedge \left(-\dot{D}K^{cd} - K^c{}_e \wedge K^{ed}\right)$$

$$\dot{R}^a{}_b + \dot{D}K^a{}_b + K^a{}_c \wedge K^c{}_b = 0$$

$$\mathcal{L} = \frac{1}{4\kappa^2}\epsilon_{abcd}\, e^a \wedge e^b \wedge \dot{R}^{cd}$$

$$S_c{}^{ed} = \frac{1}{2}\left(T_a{}^{ae}\delta^d_c - T_a{}^{ad}\delta^e_c + K^{ed}{}_c\right)$$

$$T \equiv T^c{}_{ed}\, S_c{}^{ed} \Rightarrow \mathcal{L} = \frac{1}{2\kappa^2} e \cdot T$$

$$\mathcal{L} = \frac{1}{2\kappa^2} e \mathring{R}$$

$$\dot{R}^a{}_b + \dot{D}K^a{}_b + K^a{}_c \wedge K^c{}_b = 0$$

$$\frac{\delta I}{\delta e^a_\mu} = 0$$

$$\frac{4}{e}\partial_\mu\left(e S_\sigma{}^{\mu\lambda}\right) - 4T^\sigma{}_{\mu a} S_\sigma{}^{\lambda\mu} - T e^\lambda_a = 0 \Leftrightarrow \frac{\delta I}{\delta g^{\mu\nu}} = 0$$

$$\mathring{R}_{\mu\nu} - \frac{1}{2} g_{\mu\nu}\mathring{R} = 0$$

Einstein-Cartán

$$\mathcal{L} = \frac{1}{4\kappa^2}\epsilon_{abcd}\, e^a \wedge e^b \wedge \left(R^{cd} - \dot{D}K^{cd} - K^c{}_e \wedge K^{ed}\right)$$

$$de^c + w^c{}_b \wedge e^b = T^c$$

$$dw^a{}_b + w^a{}_c \wedge w^c{}_b = R^a{}_b$$

$\nabla g = 0 \Rightarrow w^a{}_b = \dot{w}^a{}_b + K^a{}_b$

$$\dot{R}^a{}_b + \dot{D}K^a{}_b + K^a{}_c \wedge K^c{}_b = R^a{}_b$$

$$\mathcal{T}^a = \mathcal{K}^a{}_b \wedge e^b$$

Conexion de Levi-Civita

$\omega^a{}_b = \dot{\omega}^a{}_b$

$$w^a{}_b = \dot{w}^a{}_b + K^a{}_b$$

$$\mathcal{T}^a = \mathcal{K}^a{}_b \wedge e^b$$

$$\dot{R}^a{}_b + \dot{D}K^a{}_b + K^a{}_c \wedge K^c{}_b = R^a{}_b$$

$\Rightarrow$

$K^a{}_b = 0$

$T^a = 0$

$\dot{R}^a{}_b + \dot{D}K^a{}_b = R^a{}_b$

$$de^c + \dot{w}^c{}_b \wedge e^b = 0$$

$$d\dot{w}^a{}_b + \dot{w}^a{}_c \wedge \dot{w}^c{}_b = \dot{R}^a{}_b$$

$$\mathcal{L} = \frac{1}{4\kappa^2}\epsilon_{abcd}\, e^a \wedge e^d \wedge \dot{R}^{cd}$$

Apendice

$$\frac{4}{e}\partial_\mu(e S_\sigma{}^{\mu\lambda}) - 4T^\sigma{}_{\mu a} S_\sigma{}^{\lambda\mu} - T e_a^\lambda = 0$$

$$\mathring{R}_{\mu\nu} - \frac{1}{2} g_{\mu\nu}\mathring{R} = 0$$

La ecu de la izquierda es la ecuacion de campo de Einstein escrita en la base no-coordenada coordenada. La ecu de la derecha es la misma ecuacion de campo Esintein pero escrita en la base coordenda. Para ir de una a otra (tomando en cuenta la conexion de weinzembook) solo debemos transformar como se transforman las componentes de los vectores, usando vielveims V^{a}=e^{a}_{\mu}V^{\mu}

Por ahora queda pendiente

1) demostrar como parti4endo de G=0 se llega a la ecuacion de los teleparalelistas o viceversa
2) Obtener la solucion de Schwarzvhild usando las ecuaciones de los teleaparalelistas
3) Obtener la solucion de Schwarzvhild usando las ecuacion de campo einstein
4) Analizar la supuesta ruptura de simetria lorentz, se puede entender si estudiamos con mas detalle la tesis sobre formas diferenciales. Esto tiene que ver con el conteo de grados libertad

Observacion gravedad teleparalela

$$\Gamma = e\partial e + ee\omega$$

$$\left.\begin{aligned} \omega \equiv 0 &\Rightarrow \Gamma = e\partial e \\ \text{Def} &\Rightarrow \Gamma = \mathring{\Gamma} + K \end{aligned}\right\} \mathring{\Gamma} = e\partial e - K$$

$$\Gamma^{\mu}{}_{\nu\sigma} = e^{\mu}{}_{b}\,\partial_{\nu} e^{b}{}_{\sigma}$$

$$\mathring{\Gamma}^{\mu}{}_{\nu\sigma} = e^{\mu}{}_{b}\,\partial_{\nu} e^{b}{}_{\sigma} + e^{\mu}{}_{b}\, e^{c}{}_{\sigma}\, \mathring{\omega}_{\nu}{}^{b}{}_{c}$$

$$\left.\begin{aligned} \Gamma^{\mu}{}_{\nu\sigma} &= \mathring{\Gamma}^{\mu}{}_{\nu\sigma} + K^{\mu}{}_{\nu\sigma} \\ K^{\mu}{}_{\nu\sigma} &= \Gamma^{\mu}{}_{\nu\sigma} - \mathring{\Gamma}^{\mu}{}_{\nu\sigma} \end{aligned}\right\} K^{\mu}{}_{\nu\sigma} = -e^{\mu}{}_{b}\, e^{c}{}_{\sigma}\, \mathring{\omega}_{\nu}{}^{b}{}_{c}$$

$$g_{\mu\nu} = \eta_{ab}\, e^{a}{}_{\mu}\, e^{b}{}_{\nu}$$

$$\partial_{\lambda} g_{\mu\nu} = \eta_{ab}\left[(\partial_{\lambda} e^{a}{}_{\mu})\, e^{b}{}_{\nu} + e^{a}{}_{\mu}\,(\partial_{\lambda} e^{b}{}_{\nu})\right]$$

$$\Gamma^{\mu}{}_{\nu\sigma} = e^{\mu}{}_{b}\,\partial_{\nu} e^{b}{}_{\sigma} \Rightarrow \begin{aligned} \partial_{\lambda} e^{a}{}_{\mu} &= \Gamma^{\alpha}{}_{\lambda\mu}\, e^{a}{}_{\alpha} \\ \partial_{\lambda} e^{b}{}_{\nu} &= \Gamma^{\alpha}{}_{\lambda\nu}\, e^{b}{}_{\alpha} \end{aligned}$$

$$\partial_{\lambda} g_{\mu\nu} = \eta_{ab}\,\Gamma^{\alpha}{}_{\lambda\mu}\, e^{a}{}_{\alpha} e^{b}{}_{\nu} + \eta_{ab}\,\Gamma^{\alpha}{}_{\lambda\nu}\, e^{a}{}_{\mu}\, e^{b}{}_{\alpha}$$

$$\partial_{\lambda} g_{\mu\nu} = \Gamma^{\alpha}{}_{\lambda\mu}\, g_{\alpha\nu} + \Gamma^{\alpha}{}_{\lambda\nu}\, g_{\mu\alpha}$$

Procedimiento en gravedad teleparalela

1) $ds^2 = g_{\mu\nu} dx^\mu dx^\nu = \eta_{ab} e^a_\mu e^b_\nu$

Hallamos: e^a_μ

2) Hallar: $\Gamma^\mu{}_{\nu\sigma} = e^\mu_b \partial_\nu e^b_\sigma$

3) Hallar: $\mathring{\Gamma}^\mu{}_{\nu\sigma} = \frac{g}{2}(\partial g + \partial g - \partial g)$

4) Hallar: $K^\mu{}_{\nu\sigma} = \Gamma^\mu{}_{\nu\sigma} - \mathring{\Gamma}^\mu{}_{\nu\sigma}$

5) Hallar $\dot{\omega}_\nu{}^b{}_c$

$\dot{\omega}_\nu{}^b{}_c = - e^b_\mu e^\sigma_c K^\mu{}_{\nu\sigma}$

Identidad: $de^c = -\frac{1}{2} C_{\mu\nu}{}^c dx^\mu \wedge dx^\nu$

$-\frac{1}{2} e^b_\mu e^a_\nu C_{ab}{}^c = -\frac{1}{2} C_{\mu\nu}{}^c = \partial_{[\mu} e^c_{\nu]}$

$de^c = \partial_{[\mu} e^c_{\nu]} dx^\mu \wedge dx^\nu$

6) Hallar la torsión: $T^c{}_{\mu\nu}$ y $T^\sigma{}_{\mu\nu}$

$T^c = de^c \Rightarrow T^c{}_{\mu\nu} = 2\partial_{[\mu} e^c_{\nu]}$

$T^\sigma{}_{\mu\nu} = 2 e^\sigma_c \partial_{[\mu} e^c_{\nu]}$

7) Hallar: $T^{a}{}_{bc}$; $K^{a}{}_{bc}$

$$T^{a}{}_{bc} = T^{\sigma}{}_{\mu\nu}\, e^{a}_{\sigma}\, e^{\mu}_{b}\, e^{\nu}_{c}$$

$$K^{a}{}_{bc} = K^{\sigma}{}_{\mu\nu}\, e^{a}_{\sigma}\, e^{\mu}_{b}\, e^{\nu}_{c}$$

8) Hallar:

$$S_{c}{}^{ed} = \frac{1}{2}\left(T^{ae}_{a}\, \delta^{d}_{c} - T_{a}{}^{ad}\, \delta^{e}_{c} + K^{ed}{}_{c} \right)$$

9) Hallar: $S_{\sigma}{}^{\mu\lambda} = S_{a}{}^{bc}\, e^{\mu}_{b}\, e^{\lambda}_{c}$

10) Hallar: $\frac{4}{e}\, e^{a}{}_{\lambda}\, \partial_{\mu}\left(e\, S_{a}{}^{\mu\lambda}\right) - 4\, T^{\sigma}{}_{\mu\lambda}\, S_{\sigma}{}^{\lambda\mu} - T = 0$