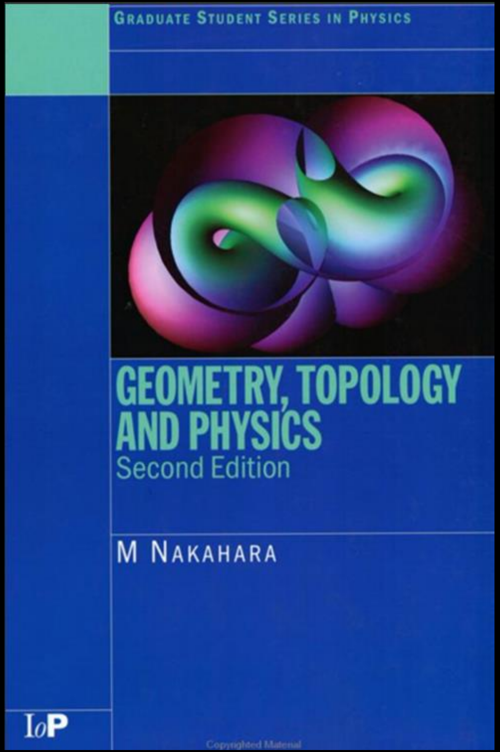
Relatividad General

2025

Prof. David Choque Q.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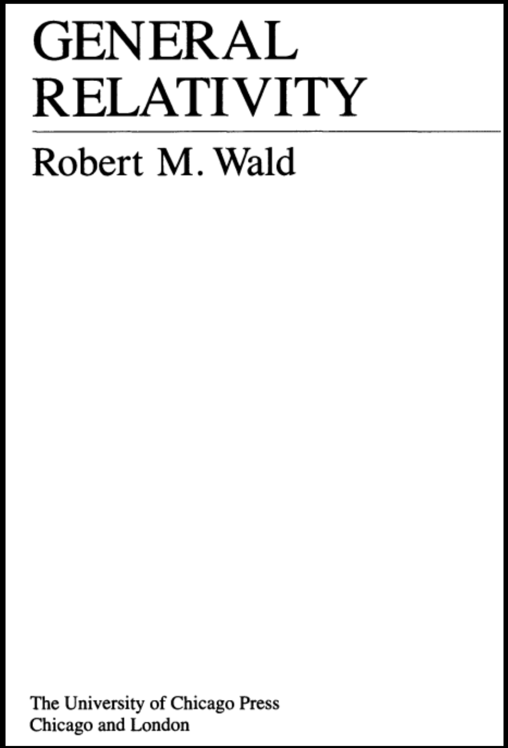
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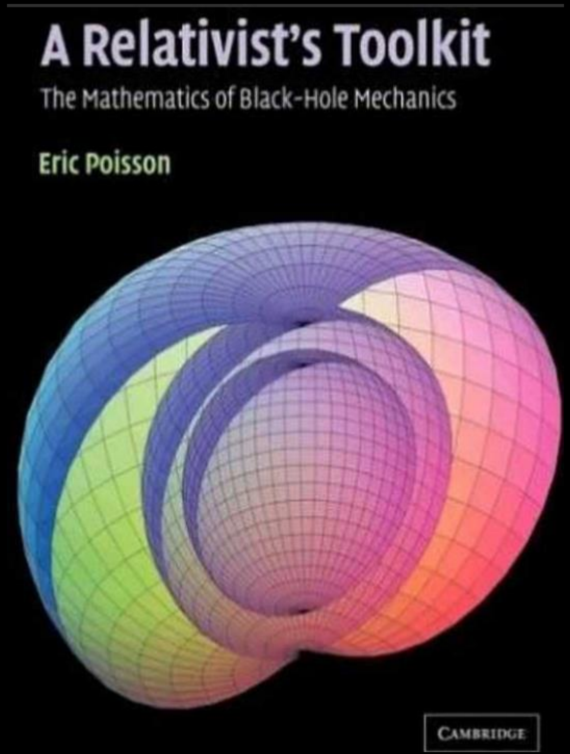
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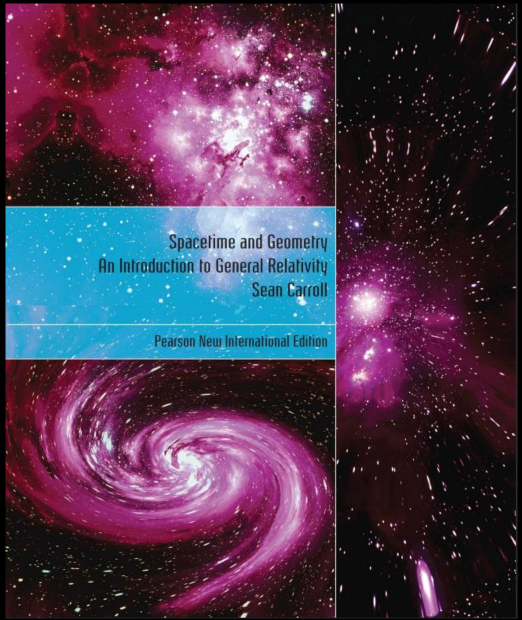
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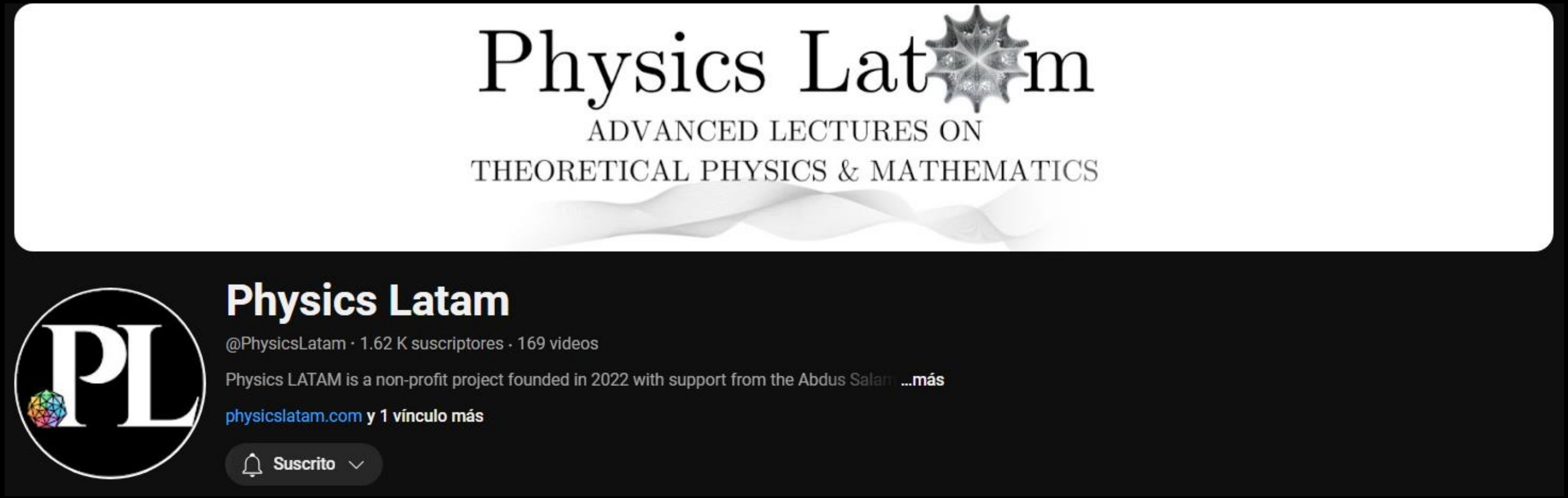
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Suscrito

Geometría de Riemann

Métrica Riemaniana

Es un tensor $(0,2) \in T_p^*M \otimes T_p^*M$

que satisface los siguientes axiomas

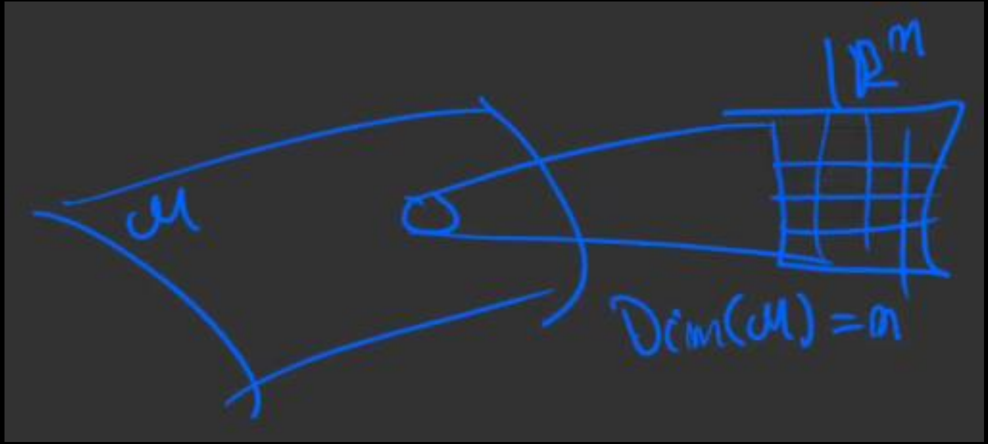
en cada $p \in M$:

Sean $U, V \in T_pM$

a) $g_p(U,V) = g_p(V,U)$

b) $g_p(U,U) \geq 0$; $g_p(U,U) = 0 \iff U = 0$

$g_p \begin{cases} \text{es simétrico positivo definido} \\ \text{es una forma bilineal} \end{cases}$



Métrica pseudo-Riemaniana

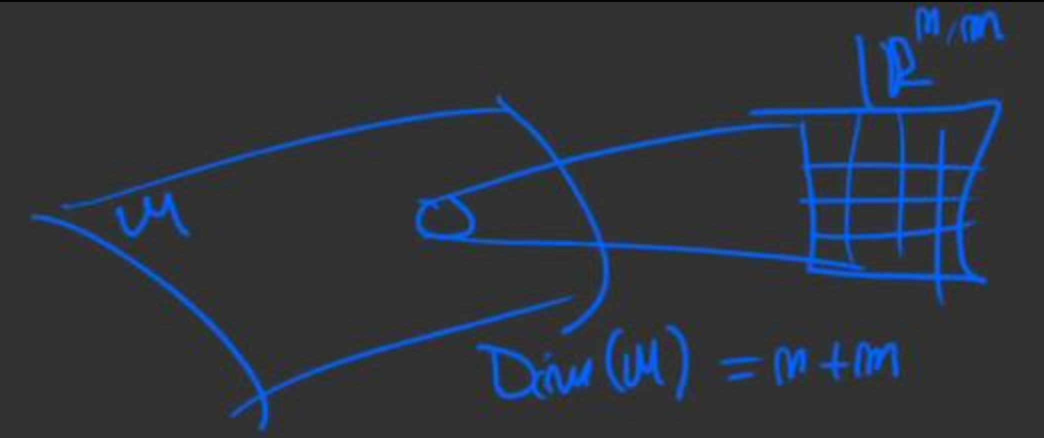
Es un tensor $(0,2) \in T_p^*M \otimes T_p^*M$ que satisface los siguientes axiomas en cada $p \in M$:

Sean $U, V \in T_pM$

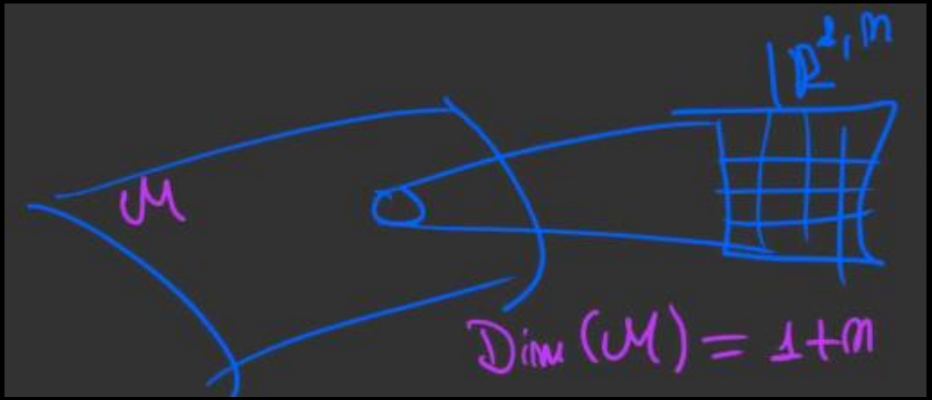
a) $g_p(U,V) = g_p(V,U)$

b') Si $g_p(U,V) = 0$ / $U \neq 0$, $V = 0$

$g_p \begin{cases} \text{es simétrico, no es positivo definido} \\ \text{es una forma bilineal} \end{cases}$



Métrica Lorentziana

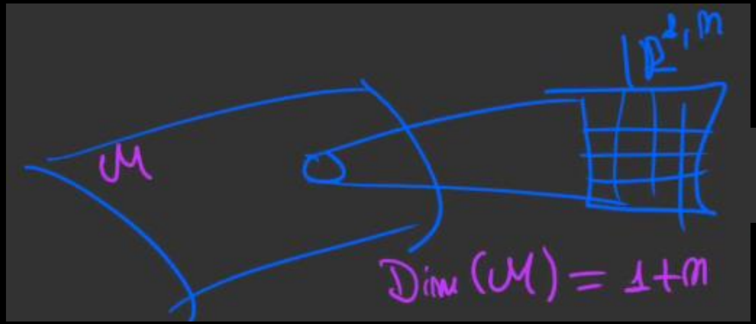


a) $g_p(U,V) = g_p(V,U)$

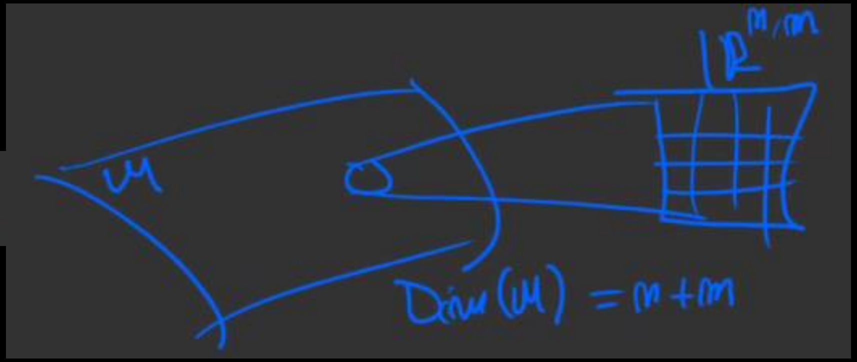
$g_p(U,U) > 0 \rightarrow U$ - tipo espacio

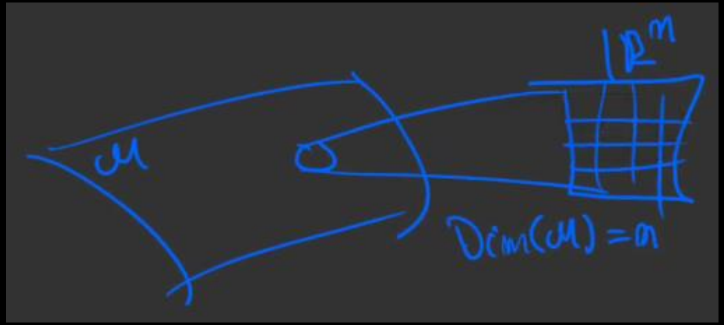
$g_p(U,U) = 0 \rightarrow U$ - tipo luz (null)

$g_p(U,U) < 0 \rightarrow U$ - tipo tiempo



$\subset$





$$g_p = g = g_{\mu\nu}\, dx^\mu \otimes dx^\nu = g_{\mu\nu}\, dx^\mu dx^\nu$$

$$g_p : T_pM \times T_pM \to \mathbb{R}$$

$$\underbrace{u\,,}_{g_p(u,\ )}\quad v \to g_p(u,v)$$

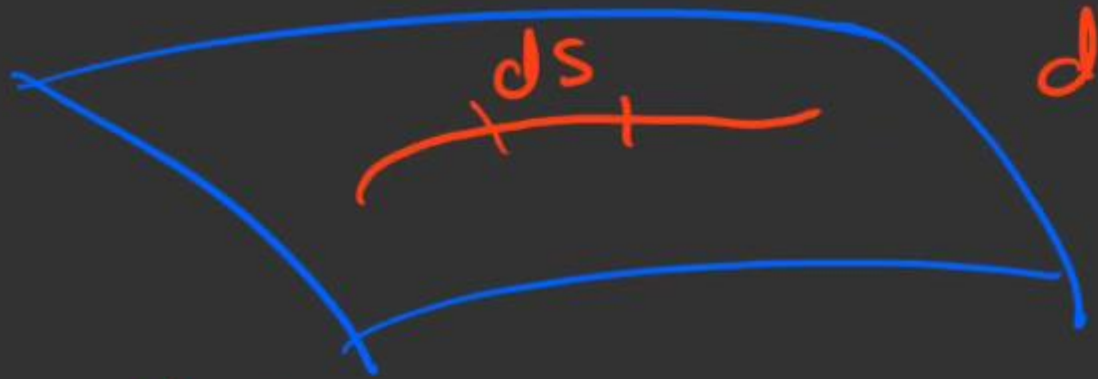
$$\langle\ ,\ \rangle : T_p^*M \times T_pM \to \mathbb{R}$$

$g_p(u,\ )$ genera un isomorfismo : T_pM y T_p^*M

$A_\mu = g_{\mu\nu} A^\nu$; $A^\mu = g^{\mu\nu} A_\nu$

$$A = A_\mu dx^\mu$$

$$A = g_{\mu\nu} A^\nu dx^\mu = A^\nu \partial_\nu$$



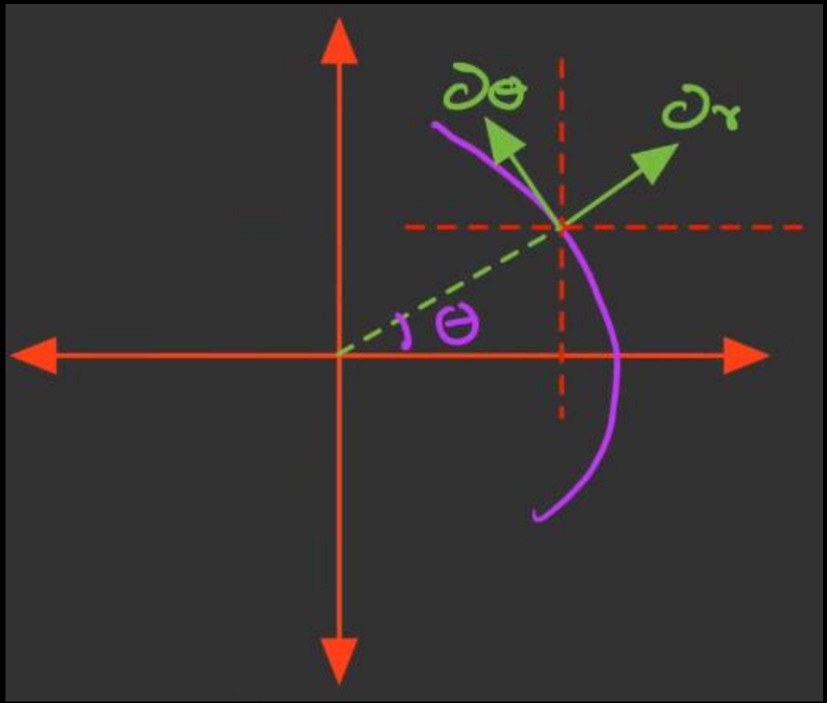
$$ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu$$

Carta $\varphi_1 : \{x^\mu\} \Rightarrow ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu$

Carta $\varphi_2 : \{x'^\mu\} \rightarrow ds'^2 = g'_{\mu\nu}\, dx'^\mu dx'^\nu$

$ds^2 = ds'^2$: $ds^2 = ds'^2$: Los leyes de la física son independientes de la carta utilizada

Transporte paralelo



$$x = r\cos\theta \qquad r(x,y)$$
$$y = r\sin\theta \quad \Rightarrow \quad \theta(x,y)$$

$$\partial_\theta = -r\sin\theta\,\partial_x + r\cos\theta\,\partial_y$$

$$\partial_r = \cos\theta\,\partial_x + \sin\theta\,\partial_y$$

$$\frac{\partial(\partial_r)}{\partial r} = 0$$

$$\frac{\partial(\partial_r)}{\partial \theta} = -\sin\theta\,\partial_x + \cos\theta\,\partial_y = r\,\partial_\theta$$

$$\frac{\partial(\partial_r)}{\partial r} = \Gamma^r\,\partial_r + \Gamma^\theta\,\partial_\theta$$

$$\frac{\partial(\partial_r)}{\partial \theta} = \Gamma^r\,\partial_r + \Gamma^\theta\,\partial_\theta$$

$$\frac{\partial(\partial_r)}{\partial r} = \Gamma^r_r\,\partial_r + \Gamma^\theta_r\,\partial_\theta$$

$$\frac{\partial(\partial_r)}{\partial \theta} = \Gamma^r_r\,\partial_r + \Gamma^\theta_r\,\partial_\theta$$

$$\frac{\partial(\partial_r)}{\partial r} = \Gamma^r_{rr}\,\partial_r + \Gamma^\theta_{rr}\,\partial_\theta = \Gamma^\mu_{rr}\,\partial_\mu$$

$$\frac{\partial(\partial_r)}{\partial \theta} = \Gamma^r_{r\theta}\,\partial_r + \Gamma^\theta_{r\theta}\,\partial_\theta = \Gamma^\mu_{r\theta}\,\partial_\mu$$

$$\frac{\partial(\partial_\nu)}{\partial x^\sigma} = \Gamma^\mu_{\nu\sigma}\,\partial_\mu$$

El simbolo de Christoffel nos dice como cambian los vectores base en cada punto de la variedad

$$\frac{\partial(\partial_\nu)}{\partial x^\sigma} = \Gamma^\mu_{\nu\sigma}\,\partial_\mu$$

El simbolo de Christoffel nos dice como cambian los vectores base en cada punto de la variedad

Sean : $X, Y \in T_p\mathcal{M}$

$\nabla : T_p\mathcal{M} \times T_p\mathcal{M} \to T_p\mathcal{M}$

$$\nabla : \mathfrak{X}(\mathcal{M}) \times \mathfrak{X}(\mathcal{M}) \to \mathfrak{X}(\mathcal{M})$$
$$(X, Y) \to \nabla_X Y$$

Conexion : ∇

Coeficientes de la conexión : $\nabla_\nu(\partial_\sigma) = \Gamma_{\nu\sigma}{}^\mu\,\partial_\mu$

$$\nabla_\mu dx^\nu = -\Gamma^\nu{}_{\mu\lambda} dx^\lambda$$

$$\nabla_X(Y+Z) = \nabla_X Y + \nabla_X Z$$

$$\nabla_{X+Y} Z = \nabla_X Z + \nabla_Y Z$$

$$\nabla_{(\phi X)} Y = \phi \nabla_X Y$$

$$\nabla : T_pM \times T_pM \to T_pM$$

$$(V \;,\; W) \to \nabla_V W$$

$$V = V^\mu \partial_\mu \;;\; W = W^\nu \partial_\nu$$

$$\nabla_V W = V^\mu \nabla_\mu W = V^\mu \nabla_\mu (W^\nu \partial_\nu)$$

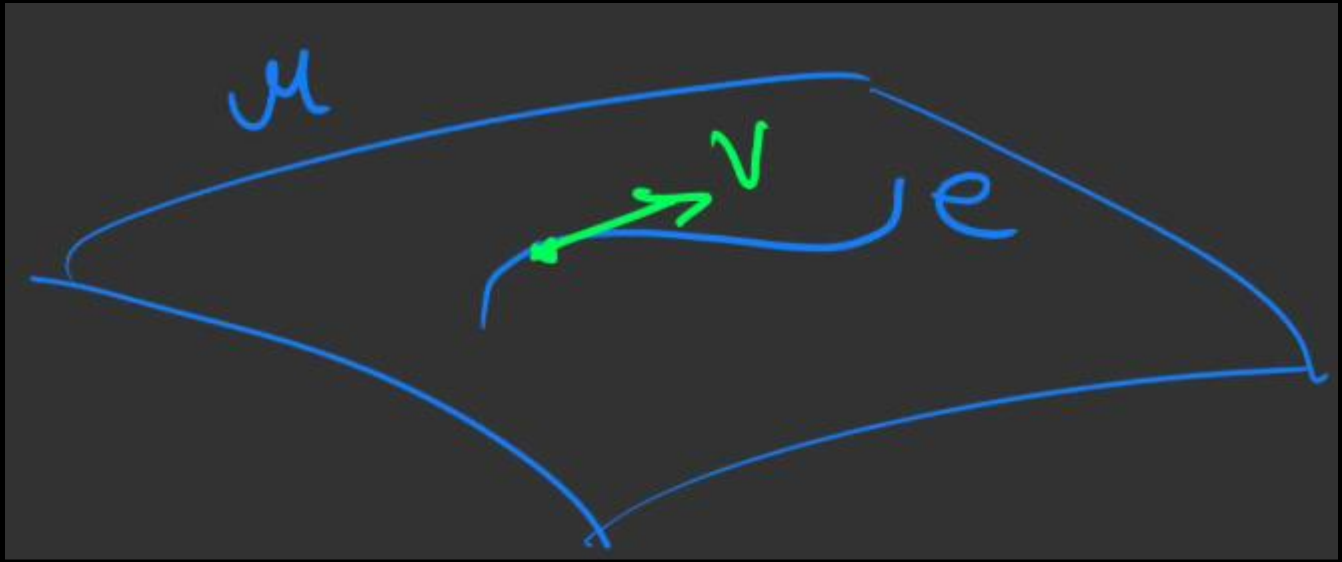
$$\nabla_V W = V^\mu (\partial_\mu W^\sigma + \Gamma^\sigma_{\mu\nu} W^\nu) \partial_\sigma$$

$$\nabla_V W = V^\mu (\nabla_\mu W^\sigma) \partial_\sigma$$

$$\nabla_\mu W^\sigma \equiv \partial_\mu W^\sigma + \Gamma^\sigma_{\mu\nu} W^\nu$$

$\nabla_V W$: Es independiente de la derivada de "V"

Transporte, transporte paralelo, geodesica



$\varphi_i =: \{x^\mu\}$

Base $T_p\mathcal{M} =: \{\partial_\mu\}$

$\mathcal{C}(t)$: curva

$(a,b) \longrightarrow \mathcal{M}$

$t \longrightarrow x^\mu(t)$

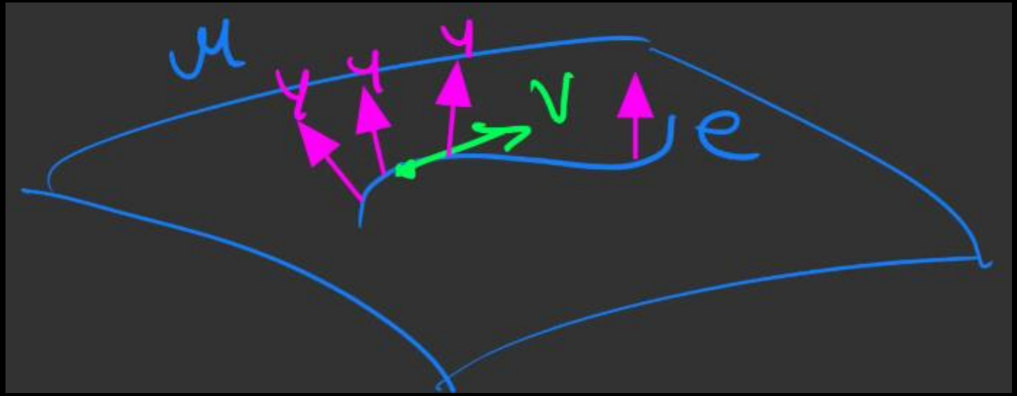
Sea: $Y \in T_p\mathcal{M}$ definida en la curva $\mathcal{C}(t)$

$$Y\big|_{\mathcal{C}(t)} = Y^\mu \partial_\mu$$

Sea: $V = \dfrac{dx^\mu}{dt} e_\mu \Big|_{\mathcal{C}(t)}$ el vector tangente a la curva $\mathcal{C}(t)$

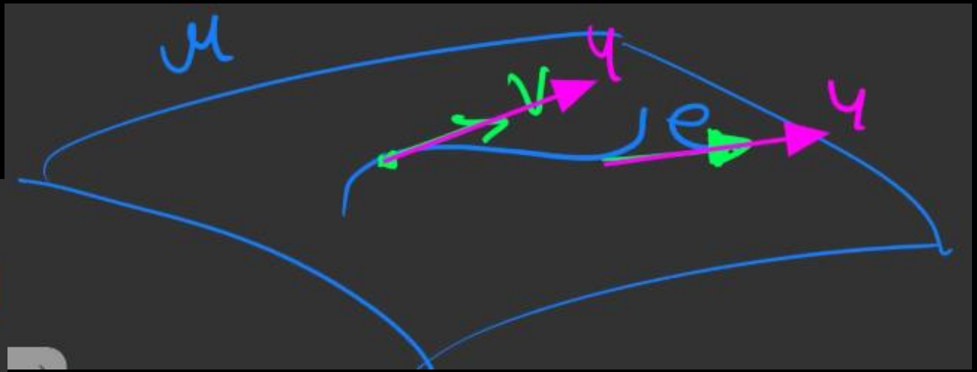
$\nabla_V Y \equiv$ Y es transportado a lo largo de $\mathcal{C}(t)$

$$\nabla_V Y = \left[\frac{dx^\mu}{dt}\frac{\partial Y^\sigma}{\partial x^\mu} + \Gamma^\sigma_{\mu\nu}\frac{dx^\mu}{dt}Y^\nu\right]\partial_\sigma$$



$\nabla_V Y = 0 \Rightarrow$ Y es transportado paralelamente a lo largo de $\mathcal{C}(t)$

$$\nabla_V Y = 0 \Rightarrow \frac{dY^\sigma}{dt} + \Gamma^\sigma_{\mu\nu}\frac{dx^\mu}{dt}Y^\nu = 0$$

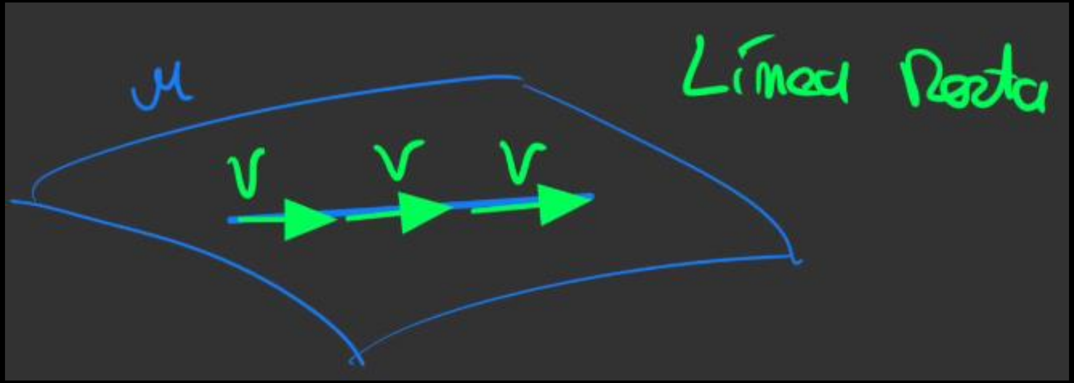


$\nabla_V V = 0 \Rightarrow$ V es transportado paralelamente a lo largo de $\mathcal{C}(t) \equiv$ Geodésica

$\Rightarrow Y \equiv V$

$Y^\nu \equiv \frac{dx^\nu}{dt}$

$$\frac{d^2x^\nu}{dt^2} + \Gamma^\mu_{\nu\lambda}\frac{dx^\mu}{dt}\frac{dx^\lambda}{dt} = 0$$



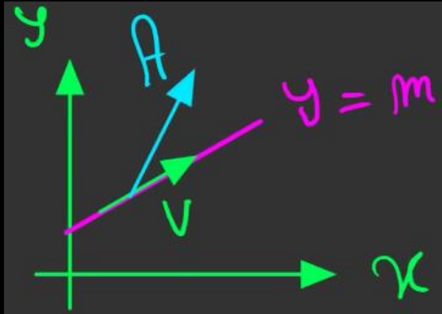
Ejemplo

$$ds^2 = dx^2 + dy^2$$

$$\mathcal{M} = \mathbb{R}^2$$

$$\varphi := \{x, y\}$$

$$\{\partial_x ; \partial_y\}$$



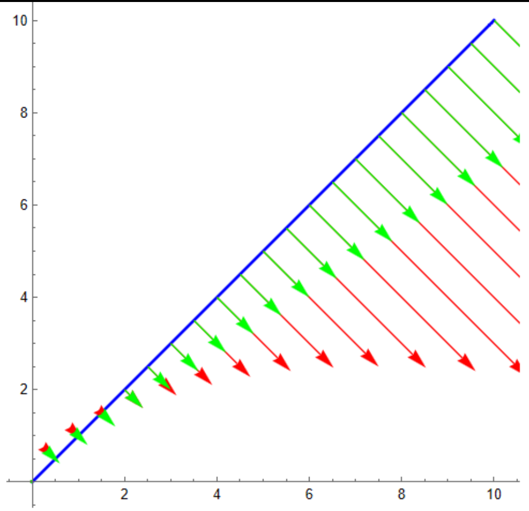
$$\Rightarrow \mathcal{C}(t) = \begin{cases} x = t \\ y = mt + b \end{cases}$$

"A" es transportado a lo largo de "v" $\nabla_V A \neq 0$

A es transportado paralelamente a lo largo de $\mathcal{C}(t)$

$$\nabla_V A = 0 \Rightarrow \frac{dA^\sigma}{dt} + \Gamma^\sigma_{\mu\nu}\frac{dx^\mu}{dt}A^\nu = 0$$

$$\frac{dA^\sigma}{dt} = 0 \quad A = A^x(x,y)\partial_x + A^y(x,y)\partial_y\big|_{\mathcal{C}(t)} = A^x(t)\partial_x + A^y(t)\partial_y$$



$$y = t \quad (m = 1 ; b = 0)$$

$$x = t$$

$$A = y^2\partial_x - x^2\partial_y$$

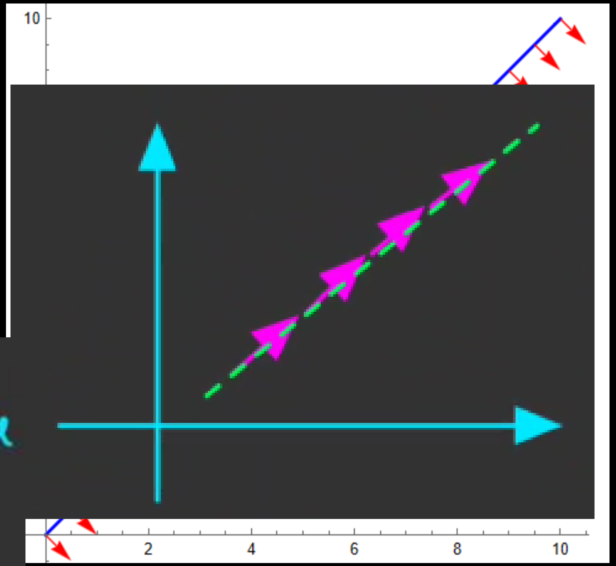
$$A = t^2\partial_x - t^2\partial_y$$

$$\nabla_V A = 2t\partial_x - 2t\partial_y$$

El transporte de A a lo largo de V: vectores verdes

$$\frac{dA^x}{dt} = 0 \Rightarrow A^x = c_1$$

$$\frac{dA^y}{dt} = 0 \Rightarrow A^y = c_2$$

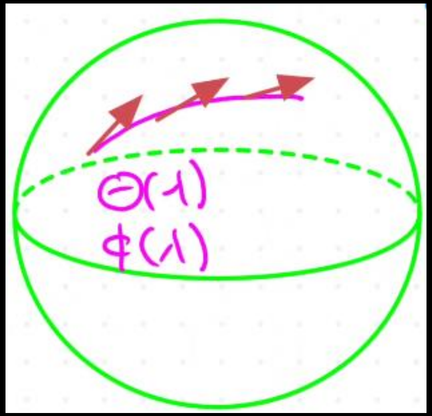


V es transportado paralelamente a lo largo de $\mathcal{C}(t) \equiv$ Geodésica

$$\frac{d^2x^\nu}{dt^2} + \Gamma^\nu_{\mu\lambda}\frac{dx^\mu}{dt}\frac{dx^\lambda}{dt} = 0$$

$$\left.\begin{aligned} \frac{d^2x}{dt^2} = 0 &\Rightarrow x = c_1 t + c_2 \\ \frac{d^2y}{dt^2} = 0 &\Rightarrow y = c_3 t + c_2 \end{aligned}\right\} \text{Recta}$$

Mundo esferico



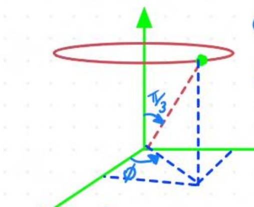
$$R = cte$$

$$ds^2 = R^2\left(d\theta^2 + \sin^2\theta \, d\phi^2\right)$$

$$g_{ij} \doteq R^2 \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{pmatrix}_{2\times 2}$$

$$\Gamma^{\theta}_{\phi\phi} = -\sin\theta\cos\theta$$

$$\Gamma^{\phi}_{\phi\theta} = \frac{\cos\theta}{\sin\theta}$$



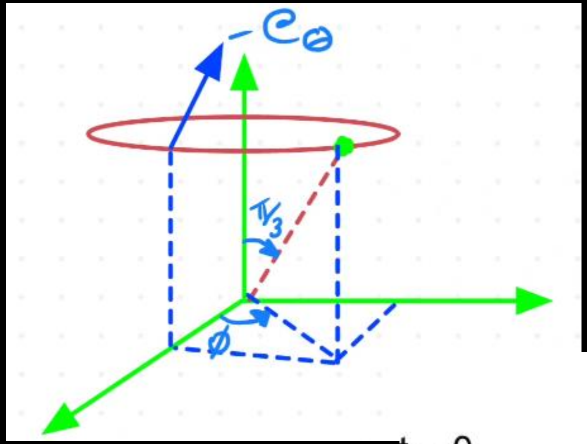
$\theta(\lambda) = \frac{\pi}{3}$

$\phi(\lambda) = \lambda$

$\lambda \in [0, 2\pi]$

$u^{\theta} = \frac{d\theta}{d\lambda} = 0$

$U^{\phi} = \frac{d\phi}{d\lambda} = 1$



Condición inicial:

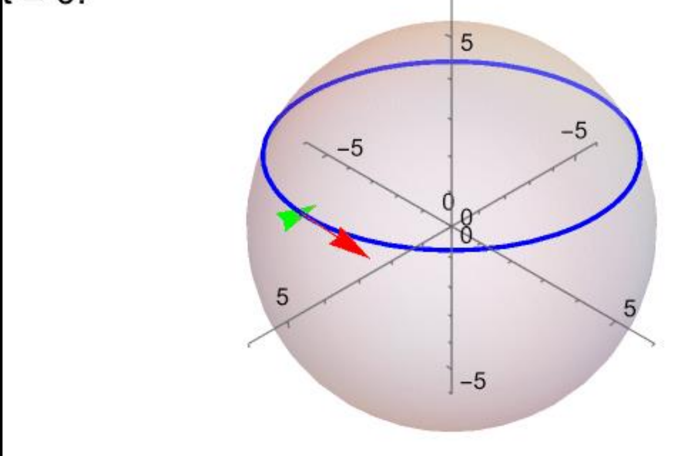
$$A\left(\theta = \frac{\pi}{3};\ \phi = 0\right) = -e_{\theta}$$

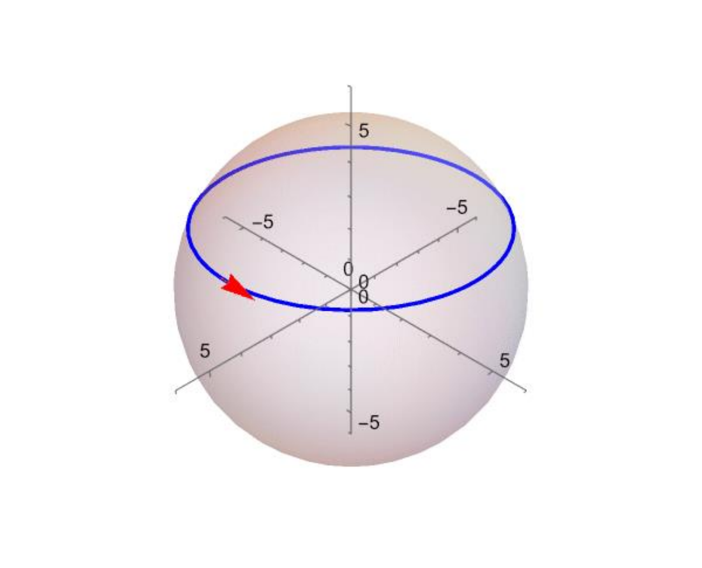
$$\nabla_u A = \left[\frac{dA^{\sigma}}{d\lambda} + \Gamma^{\sigma}_{\mu\nu} u^{\mu} A^{\nu}\right]\partial_{\sigma} = 0$$

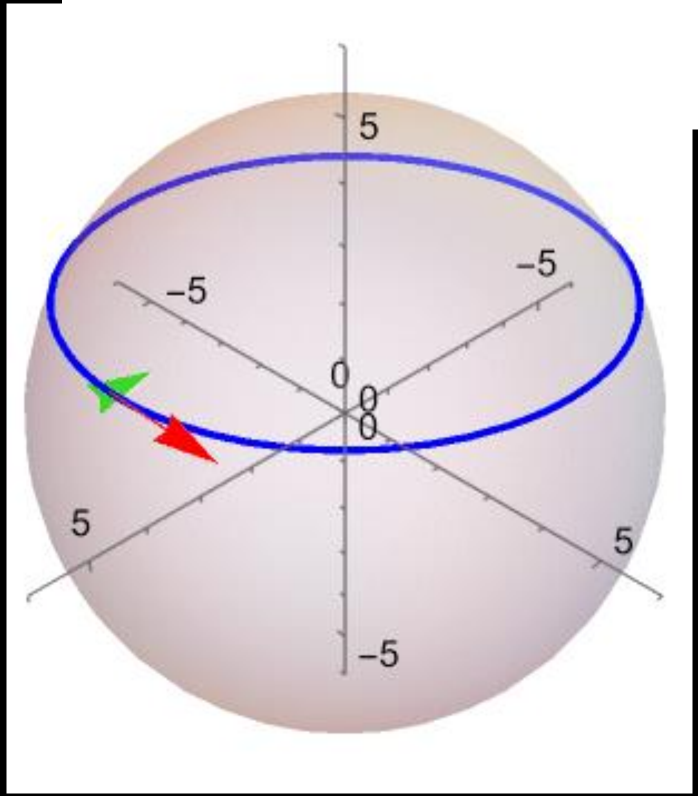
$$\partial_{\phi} A^{\theta} - \frac{\sqrt{3}}{4} A^{\phi} = 0$$

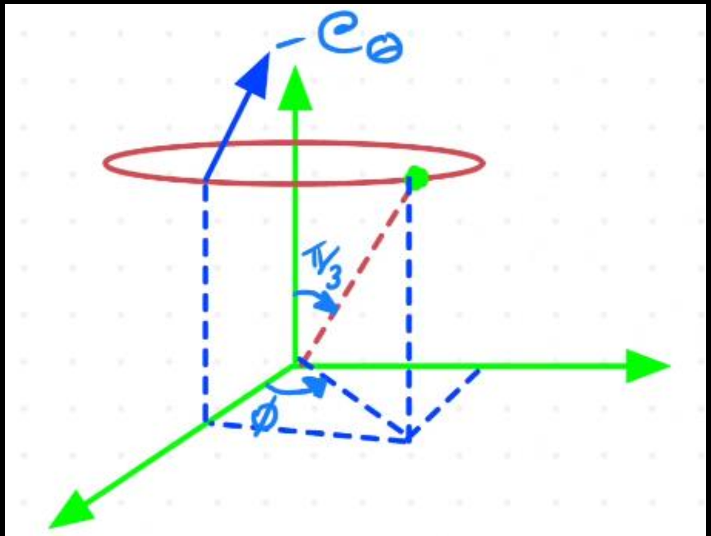
$$\partial_{\phi} A^{\phi} + \frac{1}{\sqrt{3}} A^{\theta} = 0$$

t = 0.





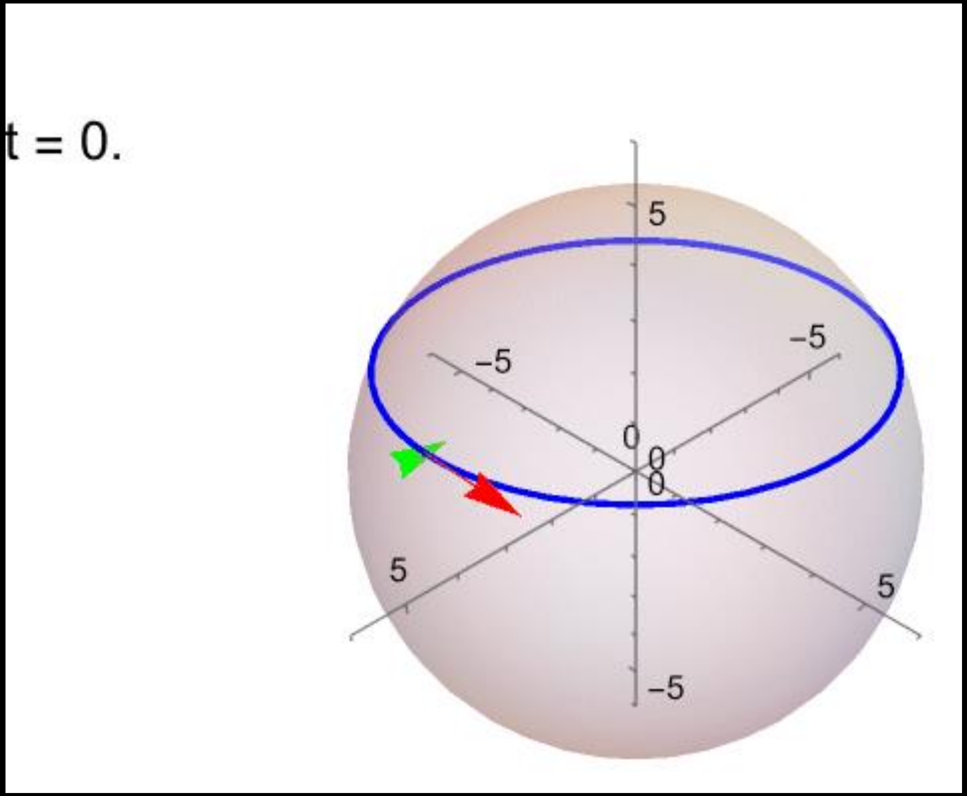


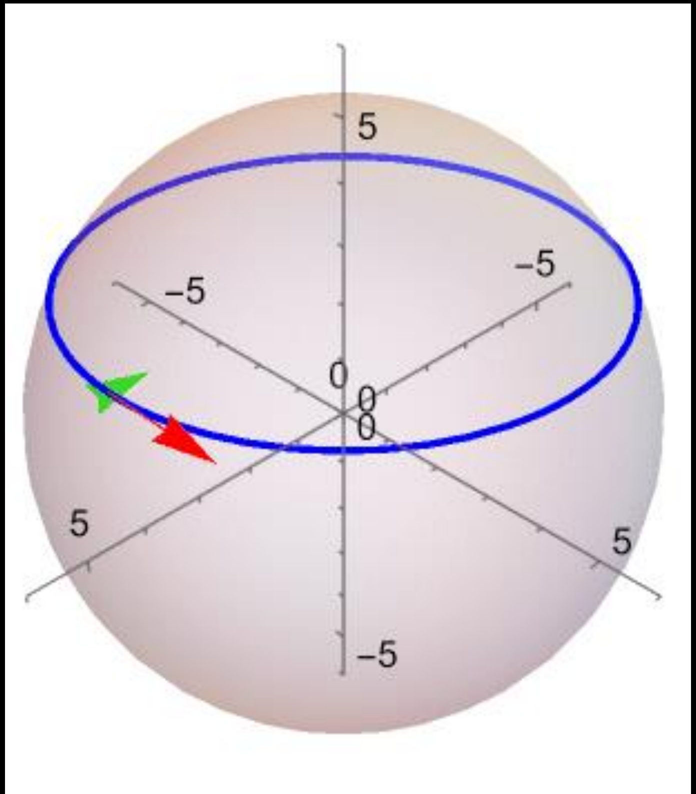


Condición inicial:

$$A\left(\theta=\frac{\pi}{3};\ \phi=0\right)=-e_\theta$$

$$A(\phi)=-\cos\left(\frac{\phi}{2}\right)e_\theta+\frac{2}{\sqrt{3}}\sin\left(\frac{\phi}{2}\right)e_\phi$$





Sea $(\mathcal{M}, g)$

$\Phi := \{x^\mu\}$

Base $T_p\mathcal{M} =: \{\partial_\mu\}$

$$\nabla : T_p\mathcal{M} \times (T_p\mathcal{M} \otimes T_p\mathcal{M}) \longrightarrow T_p\mathcal{M} \otimes T_p\mathcal{M}$$
$$(V \, , \, T) \qquad \qquad \nabla_V T$$

$$T = T^{\mu\nu} \partial_\mu \otimes \partial_\nu \in T_p\mathcal{M} \otimes T_p\mathcal{M}$$
$$V = V^\sigma \partial_\sigma \in T_p\mathcal{M}$$
$$\nabla_V T = V^\sigma \nabla_\sigma [T^{\mu\nu} \partial_\mu \otimes \partial_\nu]$$
$$= V^\sigma [(\partial_\sigma T^{\mu\nu}) \partial_\mu \otimes \partial_\nu + T^{\mu\nu} (\nabla_\sigma \partial_\mu) \otimes \partial_\nu + T^{\mu\nu} \partial_\mu \otimes \nabla_\sigma(\partial_\nu)]$$

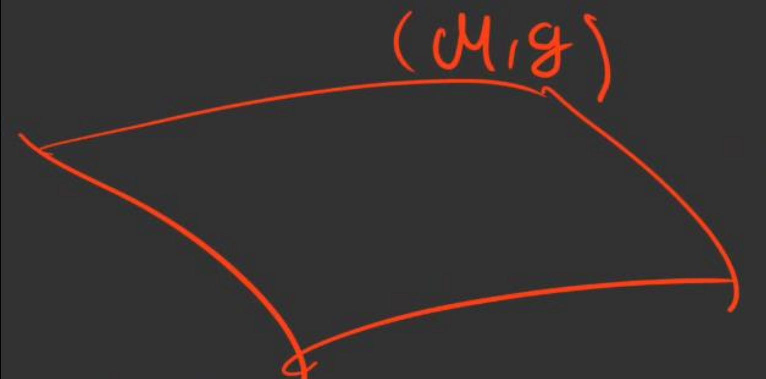
$$\nabla_V T = V^\sigma (\nabla_\sigma T^{\mu\nu}) \partial_\mu \otimes \partial_\nu$$

donde

$$\nabla_\sigma T^{\mu\nu} \equiv (\partial_\sigma T^{\mu\nu}) + \Gamma^\mu_{\sigma\beta} T^{\beta\nu} + \Gamma^\nu_{\sigma\beta} T^{\mu\beta}$$

$$(\nabla_\nu g)_{\lambda\mu} = \partial_\nu g_{\lambda\mu} - \Gamma^\kappa{}_{\nu\lambda} g_{\kappa\mu} - \Gamma^\kappa{}_{\nu\mu} g_{\lambda\kappa}$$

Transformación de los coeficientes de la conexión



$\varphi_1 =: \{x^\mu\}$

$\varphi_2 =: \{y^\mu\}$

Los coeficientes de la conexión

$\Gamma^{\alpha}_{\beta\gamma}(x)$

$\tilde{\Gamma}^{\alpha}_{\beta\gamma}(y)$

Base $\varphi_1 \Rightarrow \dfrac{\partial}{\partial x^\mu} = \partial_\mu$

Base $\varphi_2 \Rightarrow \dfrac{\partial}{\partial y^\mu} = \tilde{\partial}_\mu$

$$\tilde{\nabla}_\alpha \tilde{\partial}_\beta = \tilde{\Gamma}_{\alpha\beta}{}^{\gamma}\, \tilde{\partial}_\gamma$$

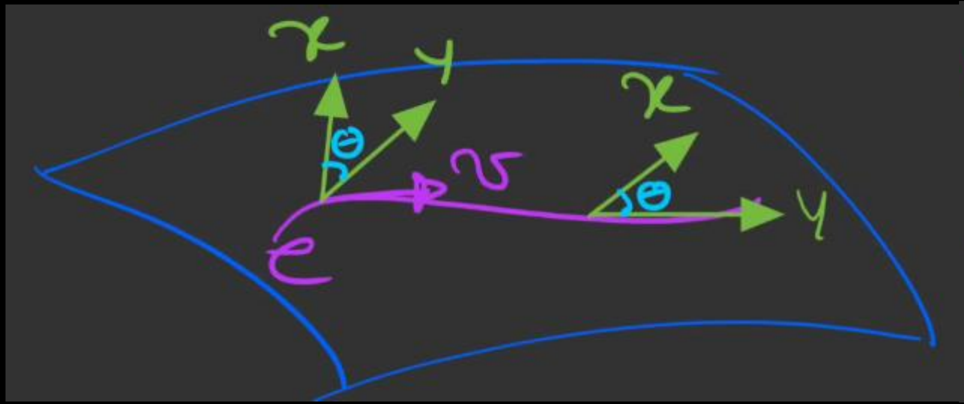
$$\tilde{\Gamma}^{\gamma}_{\alpha\beta} = \frac{\partial x^\mu}{\partial y^\beta} \frac{\partial x^\lambda}{\partial y^\alpha} \frac{\partial y^\gamma}{\partial x^\nu} \Gamma^{\nu}_{\lambda\mu} + \frac{\partial^2 x^\nu}{\partial y^\alpha \partial y^\beta} \frac{\partial y^\gamma}{\partial x^\nu}$$

*) $\Gamma^{\gamma}_{\alpha\beta}$ no son los componentes de un tensor ; Γ "no es un tensor"

*) $\dfrac{\partial^2 x^\nu}{\partial y^\alpha \partial y^\beta} = 0 \Rightarrow x^\nu = \Lambda^{\nu}{}_{\sigma}\, y^\sigma + (A_0)^\nu$ ¿Lorentz?

$$\tilde{\Gamma}^{\gamma}_{\alpha\beta} = \frac{\partial x^\mu}{\partial y^\beta} \frac{\partial x^\lambda}{\partial y^\alpha} \frac{\partial y^\gamma}{\partial x^\nu} \Gamma^{\nu}_{\lambda\mu}$$

Métrica compatible=Conexion metrica



¡Condicion geométrica Razonable!

$$\nabla_V X = 0$$

$$\nabla_V Y = 0$$

X y Y son transportados paralelamente a lo largo de la curva $\mathcal{C}$

$$g(X,Y) = g_{\mu\nu} X^{\mu} Y^{\nu} = cte$$

$$0 = \nabla_V [g(X,Y)]$$

$$0 = (\nabla_V g)(X,Y) + g(\nabla_V X, Y) + g(X, \nabla_V Y)$$

$$0 = V^{\sigma} (\nabla_{\sigma} g_{\mu\nu}) X^{\mu} Y^{\nu}$$

$$0 = V^\sigma (\nabla_\sigma g_{\mu\nu}) x^\mu y^\nu$$

Para todo: $V^\sigma \neq 0$; $x^\mu \neq 0$; $y^\nu \neq 0$

$$0 = \nabla_\sigma g_{\mu\nu}$$

$\Gamma(g, \partial g) \Rightarrow \nabla$: La conexión afín es métrica compatible "Conexión Métrica"

$$\nabla_\lambda g_{\mu\nu} = 0$$

$$\partial_\lambda g_{\mu\nu} - \Gamma^\kappa_{\lambda\mu} g_{\kappa\nu} - \Gamma^\kappa_{\lambda\nu} g_{\kappa\mu} = 0$$

$$\partial_\lambda g_{\mu\nu} - \Gamma^\kappa_{\lambda\mu} g_{\kappa\nu} - \Gamma^\kappa_{\lambda\nu} g_{\kappa\mu} = 0 \quad \ldots (a)$$

$\lambda \to \mu$; $\mu \to \nu$; $\nu \to \lambda$

$$\partial_\mu g_{\nu\lambda} - \Gamma^\kappa_{\mu\nu} g_{\kappa\lambda} - \Gamma^\kappa_{\mu\lambda} g_{\kappa\nu} = 0 \quad \ldots (b)$$

$\lambda \to \mu$; $\mu \to \nu$; $\nu \to \lambda$

$$\partial_\nu g_{\lambda\mu} - \Gamma^\kappa_{\nu\lambda} g_{\kappa\mu} - \Gamma^\kappa_{\nu\mu} g_{\kappa\lambda} = 0 \quad \ldots (c)$$

Combinación $-(a) + (b) + (c)$

$$
\begin{aligned}
&-\partial_\lambda g_{\mu\nu} + \Gamma^\kappa_{\lambda\mu}\, g_{\kappa\nu} + \Gamma^\kappa_{\lambda\nu}\, g_{\kappa\mu} \\
&+\partial_\mu g_{\nu\lambda} - \Gamma^\kappa_{\mu\nu}\, g_{\kappa\lambda} - \Gamma^\kappa_{\mu\lambda}\, g_{\kappa\nu} \\
&+\partial_\nu g_{\lambda\mu} - \Gamma^\kappa_{\nu\lambda}\, g_{\kappa\mu} - \Gamma^\kappa_{\nu\mu}\, g_{\kappa\lambda} = 0
\end{aligned}
$$

$$
\begin{aligned}
&-\partial_\lambda g_{\mu\nu} + \Gamma^\kappa_{\lambda\mu}\, g_{\kappa\nu} + \Gamma^\kappa_{\lambda\nu}\, g_{\kappa\mu} \\
&+\partial_\mu g_{\nu\lambda} - \Gamma^\kappa_{\mu\nu}\, g_{\kappa\lambda} - \Gamma^\kappa_{\mu\lambda}\, g_{\kappa\nu} \\
&+\partial_\nu g_{\lambda\mu} - \Gamma^\kappa_{\nu\lambda}\, g_{\kappa\mu} - \Gamma^\kappa_{\nu\mu}\, g_{\kappa\lambda} = 0
\end{aligned}
$$

$$
\begin{aligned}
&-\partial_\lambda g_{\mu\nu} + \partial_\mu g_{\nu\lambda} + \partial_\nu g_{\lambda\mu} \\
&+ g_{\kappa\nu}\left(\Gamma^\kappa_{\lambda\mu} - \Gamma^\kappa_{\mu\lambda}\right) - g_{\kappa\lambda}\left(\Gamma^\kappa_{\mu\nu} + \Gamma^\kappa_{\nu\mu}\right) + g_{\kappa\mu}\left(\Gamma^\kappa_{\lambda\nu} - \Gamma^\kappa_{\nu\lambda}\right) = 0
\end{aligned}
$$

$$
\Gamma^\kappa_{[\lambda\mu]} \equiv \frac{1}{2}\left(\Gamma^\kappa_{\lambda\mu} - \Gamma^\kappa_{\mu\lambda}\right) \qquad \Gamma^\kappa_{(\mu\nu)} \equiv \frac{1}{2}\left(\Gamma^\kappa_{\mu\nu} + \Gamma^\kappa_{\nu\mu}\right)
$$

$$
-\partial_\lambda g_{\mu\nu} + \partial_\mu g_{\nu\lambda} + \partial_\nu g_{\lambda\mu} - 2 g_{\kappa\lambda}\, \Gamma^\kappa_{(\mu\nu)} + 2 g_{\kappa\nu}\, \Gamma^\kappa_{[\lambda\mu]} + 2 g_{\kappa\mu}\, \Gamma^\kappa_{[\lambda\nu]} = 0
$$

Demuestre que los componentes de $\Gamma^{\kappa}{}_{[\lambda\mu]}$ se transforma como un tensor

$$\Gamma^{\kappa}{}_{[\lambda\mu]} = \frac{1}{2} T^{\kappa}{}_{\lambda\mu}$$

$T = T^{\kappa}{}_{\lambda\mu}\, \partial_\kappa \otimes dx^\lambda \otimes dx^\mu$: tensor de torsión

$$T^{\kappa}{}_{\lambda\mu} = -T^{\kappa}{}_{\mu\lambda}$$

$$-\partial_\lambda g_{\mu\nu} + \partial_\mu g_{\nu\lambda} + \partial_\nu g_{\lambda\mu} - 2 g_{\kappa\lambda} \Gamma^{\kappa}{}_{(\mu\nu)} + g_{\kappa\nu} T^{\kappa}{}_{\lambda\mu} + g_{\kappa\mu} T^{\kappa}{}_{\lambda\nu} = 0$$

$$\left[-\partial_\lambda g_{\mu\nu} + \partial_\mu g_{\nu\lambda} + \partial_\nu g_{\lambda\mu}\right] g^{\beta\lambda} + g^{\beta\lambda} g_{\kappa\nu} T^{\kappa}{}_{\lambda\mu} + g^{\beta\lambda} g_{\kappa\mu} T^{\kappa}{}_{\lambda\nu} = 2 g_{\kappa\lambda} g^{\beta\lambda} \Gamma^{\kappa}{}_{(\mu\nu)}$$

$$\frac{1}{2} g^{\beta\lambda}\left[\partial_\mu g_{\nu\lambda} + \partial_\nu g_{\lambda\mu} - \partial_\lambda g_{\mu\nu}\right] + \frac{1}{2}\left[g^{\beta\lambda} g_{\kappa\nu} T^{\kappa}{}_{\lambda\mu} + g^{\beta\lambda} g_{\kappa\mu} T^{\kappa}{}_{\lambda\nu}\right] = \delta^{\beta}_{\kappa} \Gamma^{\kappa}{}_{(\mu\nu)}$$

$$\frac{1}{2}g^{\beta\lambda}\left[\partial_\mu g_{\nu\lambda} + \partial_\nu g_{\lambda\mu} - \partial_\lambda g_{\mu\nu}\right] + \frac{1}{2}\left[g^{\beta\lambda} g_{\kappa\nu} T^{\kappa}{}_{\lambda\mu} + g^{\beta\lambda} g_{\kappa\mu} T^{\kappa}{}_{\lambda\nu}\right] = \delta^{\beta}_{\kappa}\Gamma^{\kappa}{}_{(\mu\nu)}$$

Símbolo de Christoffel

$$\left\{ {\beta \atop \mu\nu} \right\} \equiv \frac{1}{2}g^{\beta\lambda}\left[\partial_\mu g_{\nu\lambda} + \partial_\nu g_{\lambda\mu} - \partial_\lambda g_{\mu\nu}\right]$$

$$\left\{ {\beta \atop \mu\nu} \right\} + \frac{1}{2}\left(T_{\nu}{}^{\beta}{}_{\mu} + T_{\mu}{}^{\beta}{}_{\nu}\right) = \Gamma^{\kappa}{}_{(\mu\nu)}$$

$$\Gamma^{\kappa}{}_{[\lambda\mu]} \equiv \frac{1}{2}T^{\kappa}{}_{\lambda\mu}$$

$$\Gamma^{\beta}{}_{\mu\nu} = \Gamma^{\beta}{}_{(\mu\nu)} + \Gamma^{\beta}{}_{[\mu\nu]}$$

$$\Gamma^{\beta}{}_{\mu\nu} = \left\{ {\beta \atop \mu\nu} \right\} + \frac{1}{2}\left(T_{\nu}{}^{\beta}{}_{\mu} + T_{\mu}{}^{\beta}{}_{\nu}\right) + \Gamma^{\beta}{}_{[\mu\nu]}$$

$$\Gamma^{\beta}{}_{\mu\nu} = \left\{ {\beta \atop \mu\nu} \right\} + \frac{1}{2}\left(T_{\nu}{}^{\beta}{}_{\mu} + T_{\mu}{}^{\beta}{}_{\nu} + \frac{1}{2}T^{\beta}{}_{\mu\nu}\right)$$

$$\nabla_\mu g_{\sigma\tau} = 0 \;\Rightarrow\; \Gamma^\kappa{}_{\mu\nu} = \overset{\bullet}{\Gamma}{}^\kappa{}_{\mu\nu} + K^\kappa{}_{\mu\nu}$$

Conexion de Levi-Civita: $\Rightarrow \overset{\bullet}{\Gamma}{}^\kappa{}_{\mu\nu} = \frac{g^{\kappa\rho}}{2}(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\mu\nu})$

Contorsion: $K^\kappa{}_{\mu\nu} \equiv \frac{1}{2}(T^\lambda{}_{\mu\nu} + T_\mu{}^\lambda{}_\nu + T_\nu{}^\lambda{}_\mu)$

Teorema fundamental de las variedades (pseudo-)Riemanianas: Para una variedad $(\mathcal{M}, g)$ existe una unica conexion $\Gamma^\kappa{}_{\mu\nu}$ simétrica ($T = 0$, $K = 0$) compatible con $g_{\mu\nu}$, esta conexion se le conoce como **Conexion de Levi-Civita**:

$$\nabla_\mu g_{\sigma\tau} = 0 \;\Rightarrow\; \Gamma^\kappa{}_{\mu\nu} = \overset{\bullet}{\Gamma}{}^\kappa{}_{\mu\nu} = \frac{g^{\kappa\rho}}{2}(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\mu\nu})$$

Teoría de Einstein-Cartan $\Longrightarrow$ $T \equiv 0$, $K \equiv 0$ Teoría de Einstein

∇: Conexión Levi-Civita

$$R_{\mu\nu\sigma\eta}(g, \partial g, \partial^2 g),\; R_{\mu\nu}(g, \partial g, \partial^2 g),\; R(g, \partial g, \partial^2 g)$$

Tensor de torsión: T

$T : \mathfrak{X}(M) \otimes \mathfrak{X}(M) \longrightarrow \mathfrak{X}(M)$

$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]$

Tensor de curvatura de Rieman: R

$R : \mathfrak{X}(M) \otimes \mathfrak{X}(M) \otimes \mathfrak{X}(M) \longrightarrow \mathfrak{X}(M)$

$R(X,Y,Z) = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$

$R(X,Y,Z) = R(X,Y)Z$

$X = X^\lambda \partial_\lambda$

$Y = Y^\mu \partial_\mu$

$Z = Z^\nu \partial_\nu$

$R(X,Y)Z = X^\lambda Y^\mu [\nabla_\lambda \nabla_\mu - \nabla_\mu \nabla_\lambda] Z^\nu \partial_\nu$

$R(X,Y)Z = X^\lambda Y^\mu Z^\nu [\partial_\lambda \Gamma_{\mu\nu}{}^\eta - \partial_\mu \Gamma_{\lambda\nu}{}^\eta + \Gamma_{\mu\nu}{}^\sigma \Gamma_{\lambda\sigma}{}^\eta - \Gamma_{\lambda\nu}{}^\sigma \Gamma_{\mu\sigma}{}^\eta] \partial_\eta$

$R(X,Y)Z = X^\lambda Y^\mu Z^\nu R^\eta{}_{\lambda\mu\nu} \partial_\eta$

$R^\eta{}_{\lambda\mu\nu} = \partial_\lambda \Gamma_{\mu\nu}{}^\eta - \partial_\mu \Gamma_{\lambda\nu}{}^\eta + \Gamma_{\mu\nu}{}^\sigma \Gamma_{\lambda\sigma}{}^\eta - \Gamma_{\lambda\nu}{}^\sigma \Gamma_{\mu\sigma}{}^\eta$

$T(X,Y) = X^\mu Y^\nu T^\sigma{}_{\mu\nu} \partial_\sigma = (\Gamma^\sigma{}_{\mu\nu} - \Gamma^\sigma{}_{\nu\mu}) \partial_\sigma = T^\sigma{}_{\mu\nu} \partial_\sigma$

*) $T^\lambda{}_{\mu\nu} = -T^\lambda{}_{\nu\mu}$ $\qquad$ $R^\eta{}_{\lambda\mu\nu} = -R^\eta{}_{\lambda\nu\mu}$

$$R^{\eta}{}_{\lambda\mu\nu} = \partial_\lambda \Gamma_{\mu\nu}{}^{\eta} - \partial_\mu \Gamma_{\lambda\nu}{}^{\eta} + \Gamma_{\mu\nu}{}^{\sigma}\,\Gamma_{\lambda\sigma}{}^{\eta} - \Gamma_{\lambda\nu}{}^{\sigma}\,\Gamma_{\mu\sigma}{}^{\eta}$$

$$R_{\kappa\lambda\mu\nu} = g_{\kappa\sigma} R^{\sigma}{}_{\lambda\mu\nu}$$

$$R^{\kappa}{}_{\lambda\mu\nu} = - R^{\kappa}{}_{\lambda\nu\mu}$$

$$\left.\begin{aligned} R_{\kappa\lambda\mu\nu} &= - R_{\kappa\lambda\nu\mu} \\ R_{\kappa\lambda\mu\nu} &= - R_{\lambda\kappa\mu\nu} \\ R_{\kappa\lambda\mu\nu} &= R_{\mu\nu\kappa\lambda} \end{aligned}\right\}$$

Tensor de Ricci:

$$R_{\mu\nu} = R^{\kappa}{}_{\mu\kappa\nu}$$

Escalar de Ricci:

$$R = g^{\mu\nu} R_{\mu\nu} = R^{\mu}{}_{\mu} = g^{\kappa\lambda} R_{\kappa\mu\lambda\nu}$$

$$g^{\kappa\lambda} R_{\kappa\lambda\mu\nu} = 0$$

$$g^{\mu\nu} R_{\kappa\lambda\mu\nu} = 0$$

$$g^{\kappa\mu} R_{\kappa\lambda\mu\nu} = R_{\lambda\nu}$$

$$g^{\lambda\nu} R_{\kappa\lambda\mu\nu} = R_{\kappa\mu}$$

$$g^{\lambda\mu} R_{\kappa\lambda\mu\nu} = - g^{\lambda\mu} R_{\lambda\kappa\mu\nu} = - R_{\kappa\nu}$$

$$g^{\kappa\nu} R_{\kappa\lambda\mu\nu} = - g^{\kappa\nu} R_{\kappa\lambda\nu\mu} = - R_{\lambda\mu}$$

Identidades de Bianchi

1ro identidad de Bianchi

$$R(X,Y)Z + R(Z,X)Y + R(Y,Z)X = 0$$

$X = X^\lambda \partial_\lambda$

$Y = Y^\mu \partial_\mu$

$Z = Z^\nu \partial_\nu$

$$R(X,Y)Z = X^\lambda Y^\mu Z^\nu R(\partial_\lambda, \partial_\mu)\partial_\nu$$

$$R(X,Y)Z = X^\lambda Y^\mu Z^\nu R^\sigma{}_{\lambda\mu\nu}\partial_\sigma$$

$$R(X,Y)Z = X^\lambda Y^\mu Z^\nu R^\sigma{}_{\lambda\mu\nu}\partial_\sigma$$

$$R(Z,X)Y = Z^\nu X^\lambda Y^\mu R^\sigma{}_{\nu\lambda\mu}\partial_\sigma$$

$$R(Y,Z)X = Y^\mu Z^\nu X^\lambda R^\sigma{}_{\mu\nu\lambda}\partial_\sigma$$

$$X^\lambda Y^\mu Z^\nu \left[R^\sigma{}_{\lambda\mu\nu} + R^\sigma{}_{\nu\lambda\mu} + R^\sigma{}_{\mu\nu\lambda}\right]\partial_\sigma = 0 \Rightarrow R^\sigma{}_{\lambda\mu\nu} + R^\sigma{}_{\nu\lambda\mu} + R^\sigma{}_{\mu\nu\lambda} = 0$$

2do identidad de Bianchi

$$(\nabla_X R)(Y,Z)V + (\nabla_Z R)(X,Y)V + (\nabla_Y R)(Z,X)V = 0$$

$$\nabla_\kappa R^{\sigma}{}_{\lambda\mu\nu} + \nabla_\mu R^{\sigma}{}_{\lambda\nu\kappa} + \nabla_\nu R^{\sigma}{}_{\lambda\kappa\mu} = 0$$

$$\nabla_\kappa R_{\lambda\nu} + \nabla_\mu R^{\mu}{}_{\lambda\nu\kappa} - \nabla_\nu R_{\lambda\kappa} = 0$$

$$g^{\lambda\nu}\left(\nabla_\kappa R_{\lambda\nu} + \nabla_\mu R^{\mu}{}_{\lambda\nu\kappa} - \nabla_\nu R_{\lambda\kappa}\right) = 0$$

$$\nabla_\kappa R - \nabla_\mu R^{\mu}{}_{\kappa} - \nabla_\nu R^{\nu}{}_{\kappa} = 0$$

$$\nabla_\kappa R - 2\nabla_\mu R^{\mu}{}_{\kappa} = 0$$

$$\nabla_\mu \delta^{\mu}_{\kappa} R - 2\nabla_\mu R^{\mu}{}_{\kappa} = 0$$

$$-2\nabla_\mu\left[R^\mu{}_\kappa - \frac{1}{2}\delta^\mu_\kappa R\right] = 0$$

$$\nabla_\mu\left[R^{\mu\kappa} - \frac{1}{2}g^{\mu\kappa}R\right] = 0$$

$$G^{\mu\kappa} \equiv R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R$$

$$\nabla_\mu G^{\mu\kappa} = 0$$

$$G^{\mu\nu} = \frac{8\pi G_N}{c^4}T^{\mu\nu}$$

$$\nabla_\mu G^{\mu\nu} = \frac{8\pi G_N}{c^4}\nabla_\mu T^{\mu\nu}$$

$$\nabla_\mu G^{\mu\nu} = 0 \Rightarrow \nabla_\mu T^{\mu\nu} = 0$$