The Lyra-Schwarzschild Spacetime

Joel Antony Paquiyauri Ruiz

Instituto de Física Teórica - Universidade Estadual Paulista (IFT-UNESP)
Co-authors: B. M. Pimentel, M. C. Bertin, and R. R. Cuzinatto.

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A natural extension of GR consists of scalar-tensor theories, where a scalar degree of freedom complements the metric field. Well-known examples include Kaluza–Klein and Brans–Dicke theories.

In this context, Lyra proposed a generalization of Riemannian geometry by introducing a scale function ϕ into the definition of the reference frame. Lyra geometry enlarges spacetime transformations to include both diffeomorphisms and scale transformations, modifying the notions of connection, curvature, and geodesics.

Lyra Scalar-Tensor Gravity

Early formulations of Lyra gravity suffered from conceptual and variational inconsistencies or reduced to known scalar-tensor models. Recently, a consistent scalar-tensor theory on Lyra's manifold (Lyra Scalar-Tensor gravity, LyST) was constructed from symmetry principles.

In this work, we:

- Derive the field equations from a well-defined action.
- Construct the spherically symmetric (Lyra-Schwarzschild) solution.
- Study massive and massless particle motion via the Hamilton–Jacobi formalism.

Lyra Manifolds

Let $\mathcal{M}$ be a topological manifold, then:

- A *Lyra reference system* (LRS) is a triple $(\mathcal{U}, \chi, \Phi)$, where $(\mathcal{U}, \chi)$ is a chart of the manifold and $\Phi : \mathcal{U} \to \mathbb{R}^*$ is a continuous map, called *scale map*. Here, $\mathbb{R}^*$ denotes the set of all positive real numbers.
- A *Lyra atlas* is a collection of Lyra reference systems, $\mathcal{A} = \{(\mathcal{U}_\alpha, \chi_\alpha, \Phi_\alpha)\}_{\alpha \in I}$, that covers $\mathcal{M}$.
- The collection $\mathcal{A}$ is said to be a *smooth Lyra atlas* if any two Lyra reference systems in $\mathcal{A}$ are smoothly compatible and scale maps are smooth.
- The *maximal extension* of a smooth Lyra atlas $\mathcal{A}$ is the collection of all Lyra reference systems smoothly compatible with every element of $\mathcal{A}$. This is called a *maximal smooth Lyra atlas*.
- A *smooth Lyra manifold* of dimension n is a pair $(\mathcal{M}, \mathcal{A})$, where $\mathcal{M}$ is an n-dimensional topological manifold and $\mathcal{A}$ is a maximal smooth Lyra atlas on $\mathcal{M}$.

Tensors

Given a LRS $(\mathcal{U}, \chi, \Phi)$, the associated Lyra vector basis and its dual are defined, respectively, as

$$e_\mu = \frac{1}{\phi(x)} \frac{\partial}{\partial x^\mu} \quad \text{and} \quad e^\mu = \phi(x) dx^\mu. \tag{1}$$

The commutation of these vectors, $[e_\mu, e_\nu] = \gamma^\lambda{}_{\mu\nu} e_\lambda$, determines the corresponding *structure coefficients*:

$$\gamma^\lambda{}_{\mu\nu} = \phi^{-2}(\delta^\lambda_\mu \partial_\nu \phi - \delta^\lambda_\nu \partial_\mu \phi). \tag{2}$$

The transformation rules for base vectors and covectors are given by

$$e_\mu \to \bar{e}_\mu = \frac{\phi\,(x)}{\bar{\phi}\,(\bar{x})} \frac{\partial x^\nu}{\partial \bar{x}^\mu} e_\nu, \quad e^\mu \to \bar{e}^\mu = \frac{\bar{\phi}\,(\bar{x})}{\phi\,(x)} \frac{\partial \bar{x}^\mu}{\partial x^\nu} e^\nu. \tag{3}$$

This implies the following transformation rule for tensor field components:

$$\bar{T}^{\mu_1 \cdots \mu_r}{}_{\nu_1 \cdots \nu_s}(\bar{x}) = \left(\frac{\bar{\phi}(\bar{x})}{\phi(x)}\right)^{r-s} \left(\frac{\partial \bar{x}^{\mu_1}}{\partial x^{\alpha_1}} \cdots \frac{\partial \bar{x}^{\mu_r}}{\partial x^{\alpha_r}} \frac{\partial x^{\beta_1}}{\partial \bar{x}^{\nu_1}} \cdots \frac{\partial x^{\beta_s}}{\partial \bar{x}^{\nu_s}}\right) T^{\alpha_1 \cdots \alpha_r}{}_{\beta_1 \cdots \beta_s}(x),$$

Geometric Structure

The metric tensor is introduced as a non-degenerate symmetric tensor field

$$g : \Gamma(T\mathcal{M}) \times \Gamma(T\mathcal{M}) \to C^{\infty}(\mathcal{M}). \tag{4}$$

The line element in a Lyra basis is written as:

$$ds^2 = \phi^2 g_{\mu\nu} dx^{\mu} dx^{\nu} \tag{5}$$

From here, one obtains the geodesic equation:

$$\frac{d^2x^{\mu}}{dt^2} + \left\{ {\mu \atop \alpha\beta} \right\} \frac{dx^{\alpha}}{dt} \frac{dx^{\beta}}{dt} + \left(\delta^{\mu}_{\alpha} \nabla_{\beta}\phi + \delta^{\mu}_{\beta} \nabla_{\alpha}\phi - g_{\alpha\beta} g^{\mu\nu} \nabla_{\nu}\phi \right) \frac{dx^{\alpha}}{dt} \frac{dx^{\beta}}{dt} = 0, \tag{6}$$

and the volume element:

$$dV = \phi^n(x) \sqrt{\left|\det(g)\right|}\, d^n x. \tag{7}$$

A connection ∇ is defined in the usual way as an operator acting upon two vector fields. Then, one of the entries is extended to tensor fields, thus giving the *covariant derivative*:

$$\begin{aligned}\nabla_\lambda T^{\mu_1\cdots\mu_r}{}_{\nu_1\cdots\nu_s} = {} & e_\lambda T^{\mu_1\cdots\mu_r}{}_{\nu_1\cdots\nu_s} \\ & + \Gamma^{\mu_1}{}_{\sigma\lambda} T^{\sigma\cdots\mu_r}{}_{\nu_1\cdots\nu_s} + \cdots + \Gamma^{\mu_r}{}_{\sigma\lambda} T^{\mu_1\cdots\sigma}{}_{\nu_1\cdots\nu_s} \\ & - \Gamma^{\sigma}{}_{\nu_1\lambda} T^{\mu_1\cdots\mu_r}{}_{\sigma\cdots\nu_s} - \cdots - \Gamma^{\sigma}{}_{\nu_s\lambda} T^{\mu_1\cdots\mu_r}{}_{\nu_1\cdots\sigma}. \end{aligned} \tag{8}$$

Here,

$$\nabla_\lambda T^{\mu_1\cdots\mu_r}{}_{\nu_1\cdots\nu_s} := \left(\nabla_{e_\lambda} T\right)^{\mu_1\cdots\mu_r}{}_{\nu_1\cdots\nu_s},$$

and $\Gamma^\lambda{}_{\mu\nu}$ are the *connection coefficients*, which fully determine the connection.

The auto-parallel curve equation is:

$$\frac{d^2x^\mu}{dt^2} + \left(\phi\Gamma^\mu{}_{\alpha\beta} + \delta^\mu_\alpha \nabla_\beta\phi\right)\frac{dx^\alpha}{dt}\frac{dx^\beta}{dt} = 0. \tag{9}$$

Torsion:

$$\tau^{\alpha}{}_{\mu\nu} = \Gamma^{\alpha}{}_{\nu\mu} - \Gamma^{\alpha}{}_{\mu\nu} - \gamma^{\alpha}{}_{\mu\nu}. \tag{10}$$

Curvature:

$$R^{\alpha}{}_{\beta\mu\nu} = e_{\mu}\Gamma^{\alpha}{}_{\beta\nu} + \Gamma^{\alpha}{}_{\lambda\mu}\Gamma^{\lambda}{}_{\beta\nu} - e_{\nu}\Gamma^{\alpha}{}_{\beta\mu} - \Gamma^{\alpha}{}_{\lambda\nu}\Gamma^{\lambda}{}_{\beta\mu} - \Gamma^{\alpha}{}_{\beta\lambda}\gamma^{\lambda}{}_{\mu\nu}. \tag{11}$$

First Bianchi identity:

$$R^{\alpha}{}_{[\beta\mu\nu]} = \nabla_{[\beta}\tau^{\alpha}{}_{\mu\nu]} + \tau^{\alpha}{}_{\lambda[\beta}\tau^{\lambda}{}_{\mu\nu]}. \tag{12}$$

Second Bianchi identity:

$$R^{\alpha}{}_{\beta[\mu\nu;\rho]} + R^{\alpha}{}_{\beta\lambda[\rho}\tau^{\lambda}{}_{\mu\nu]} = 0. \tag{13}$$

Non-metricity tensor:

$$M_{\alpha\mu\nu} = \nabla_{\alpha}g_{\mu\nu} - \nabla_{\mu}g_{\alpha\nu} - \nabla_{\nu}g_{\mu\alpha}. \tag{14}$$

General expression for the connection coefficients:

$$\Gamma^{\lambda}{}_{\mu\nu} = \phi^{-1}\left\{ {\lambda \atop \mu\nu} \right\} + \frac{1}{2}g^{\lambda\alpha}(\gamma_{\mu\alpha\nu} + \gamma_{\nu\alpha\mu} - \gamma_{\alpha\mu\nu}) + \frac{1}{2}M^{\lambda}{}_{\mu\nu} + \frac{1}{2}g^{\lambda\alpha}(\tau_{\mu\alpha\nu} + \tau_{\nu\alpha\mu} - \tau_{\alpha\mu\nu}). \tag{15}$$

LyST Connection

Some preliminary considerations:

- Equivalence between geodesic curves and auto-parallel curves.
- Existence of a generalized divergence theorem on Lyra manifolds.
- Preservation of the scalar product of parallel transported vector fields.
- Absence of spin sources or fermionic matter fields.

This leads to

$$\nabla_\lambda g_{\mu\nu} = 0 \quad \text{and} \quad \tau^{\lambda}{}_{\mu\nu} = 0. \tag{16}$$

Then, the expression of the connection coefficients reduces to

$$\Gamma^{\mu}{}_{\alpha\beta} = \phi^{-1}\left\{ {\mu \atop \alpha\beta} \right\} + \phi^{-1}\left(\delta^{\mu}_{\beta}\nabla_\alpha\phi - g_{\alpha\beta}\nabla^{\mu}\phi\right). \tag{17}$$

From here on, the connection will be assumed to have this form, and it will be referred to as the *LyST connection*.

Curvature:

$$R^{\alpha}{}_{\beta\mu\nu} = \phi^{-2}\mathcal{R}^{\alpha}{}_{\beta\mu\nu} + \phi^{-2}\left(\delta^{\alpha}_{\mu} g_{\beta\nu} - \delta^{\alpha}_{\nu} g_{\beta\mu}\right)\nabla^{\lambda}\phi\nabla_{\lambda}\phi$$
$$+ \phi^{-1}\left(\delta^{\alpha}_{\nu}\nabla_{\mu}\nabla_{\beta}\phi - \delta^{\alpha}_{\mu}\nabla_{\nu}\nabla_{\beta}\phi + g_{\beta\mu}\nabla_{\nu}\nabla^{\alpha}\phi - g_{\beta\nu}\nabla_{\mu}\nabla^{\alpha}\phi\right). \quad (18)$$

Ricci tensor:

$$R_{\beta\nu} = \phi^{-2}\mathcal{R}_{\beta\nu} + (n-1)\phi^{-2} g_{\beta\nu}\nabla^{\lambda}\phi\nabla_{\lambda}\phi$$
$$- (n-2)\phi^{-1}\nabla_{\nu}\nabla_{\beta}\phi - \phi^{-1} g_{\beta\nu}\nabla^{\lambda}\nabla_{\lambda}\phi. \quad (19)$$

Ricci scalar:

$$R = \phi^{-2}\mathcal{R} + n(n-1)\phi^{-2}\nabla^{\lambda}\phi\nabla_{\lambda}\phi - 2(n-1)\phi^{-1}\nabla^{\lambda}\nabla_{\lambda}\phi. \quad (20)$$

Simmetries:

$$R_{\alpha\beta\mu\nu} = -R_{\alpha\beta\nu\mu}, \quad R_{\alpha\beta\mu\nu} = -R_{\beta\alpha\mu\nu}, \quad R_{\alpha\beta\mu\nu} = R_{\mu\nu\alpha\beta}. \quad (21)$$

Minimal Coupling Prescription

Given a matter field ψ_a in a Minkowski background with metric $\eta_{\mu\nu}$, the corresponding lagrangian density is extended to a Lyra spacetime through the substitution

$$\mathcal{L}_M\left(\eta_{\mu\nu}, \psi_a, \partial_\alpha \psi_a\right) \quad \rightarrow \quad \mathcal{L}_M\left(g_{\mu\nu}, \psi_a, \nabla_\alpha \psi_a\right). \tag{22}$$

Here, the scale function ϕ is naturally incorporated through the connection, ensuring that the resulting Lagrangian density remains invariant under general Lyra transformations.

The stationary-action principle $\delta S_M = 0$, with the boundary condition $\delta \psi_a = 0$, leads to the Euler-Lagrange field equations for the matter field:

$$\frac{\partial \mathcal{L}_M}{\partial \psi_a} - \nabla_\mu \frac{\partial \mathcal{L}_M}{\partial \nabla_\mu \psi_a} = 0. \tag{23}$$

Field Equations

The gravitational action in a four-dimensional Lyra spacetime is taken to be

$$S_G = \frac{1}{2\kappa} \int_V d^4x\, \phi^4 \sqrt{|\det(g)|} R, \tag{24}$$

The gravitational field equations follow from $\delta S = 0$ under variations of $g_{\mu\nu}$ and ϕ, with $\delta g_{\mu\nu}$, $\delta\phi$ and $\delta\nabla_\mu\phi$ vanishing on the boundary ∂V.

The resulting equations are

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \kappa T_{\mu\nu} \qquad \text{and} \qquad R = \kappa M, \tag{25}$$

where

$$T_{\mu\nu} := -2\frac{\partial \mathcal{L}_M}{\partial g^{\mu\nu}} + g_{\mu\nu}\mathcal{L}_M, \tag{26}$$

and

$$M = -4\mathcal{L}_M + \frac{\partial \mathcal{L}_M}{\partial \nabla_\mu \phi}\nabla_\mu\phi - \phi\left(\frac{\partial \mathcal{L}_M}{\partial \phi} - \nabla_\mu \frac{\partial \mathcal{L}_M}{\partial \nabla_\mu \phi}\right). \tag{27}$$

Note that there is a constraint:

$$M = -T. \tag{28}$$

Newtonian Limit

Consider:

- Non-relativistic velocities: $|\frac{dx^i}{dt}| \ll 1$.
- Static fields: $\partial_0 g_{\mu\nu} = 0$ and $\partial_0 \phi = 0$.
- Weak fields: $g_{\mu\nu} \approx \eta_{\mu\nu} + h_{\mu\nu}$ and $\phi = 1 + \delta\phi$, with $|h_{\mu\nu}| \ll 1$ and $|\delta\phi| \ll 1$.

The resulting equation of motion is

$$\frac{d^2\vec{x}}{dt^2} \approx -\nabla U, \tag{29}$$

and the gravitational field equation reduces to

$$\nabla^2 U \approx \frac{\kappa}{2}\rho \quad \Longrightarrow \quad \kappa = 8\pi, \tag{30}$$

where U is the *effective Newtonian potential*:

$$U := \frac{1}{2} h_{00} + \delta\phi. \tag{31}$$

In spherical coordinates, the corresponding Killing vectors are

$$\begin{aligned} \xi^{\mu}_{(1)} &= (0, 0, -\sin\varphi, -\cot\theta\cos\varphi), \\ \xi^{\mu}_{(2)} &= (0, 0, \cos\varphi, -\cot\theta\sin\varphi), \\ \xi^{\mu}_{(3)} &= (0, 0, 0, 1). \end{aligned} \tag{32}$$

The symmetry constraints for the scale function and the metric tensor are

$$\xi^{\mu}_{(i)}\partial_{\mu}\phi = 0 \qquad \text{and} \qquad \xi^{\alpha}_{(i)}\partial_{\alpha}g_{\mu\nu} + \partial_{\mu}\xi^{\alpha}_{(i)}g_{\alpha\nu} + \partial_{\nu}\xi^{\alpha}_{(i)}g_{\mu\alpha} = 0. \tag{33}$$

This yields the expression

$$ds^2 = \phi^2(t,r)\left[g_{00}(t,r)\,dt^2 + 2g_{01}(t,r)\,dtdr + g_{11}(t,r)\,dr^2 + g_{22}(t,r)\,d\Omega^2\right],$$

and, by performing convenient transformations, it is reduced to

$$ds^2 = \phi^2(t,r)\left[A(t,r)\,dt^2 - B(t,r)\,dr^2 - r^2\left(d\theta^2 + \sin^2\theta\,\varphi^2\right)\right]. \tag{34}$$

Explicit Equations

Write the undetermined functions in exponential form

$$\phi(t,r) = e^{\gamma(t,r)}, \quad A(t,r) = e^{\alpha(t,r)}, \quad B(t,r) = e^{\beta(t,r)}, \tag{35}$$

and use Cartan's procedure. The resulting equations are:

$$\frac{1}{r^2} + \left(3\dot{\gamma}^2 + 2\dot{\gamma}\dot{\beta}\right) e^{-2\alpha} - \left(2\gamma'' + \gamma'^2 - 2\gamma'\beta' + \frac{4\gamma'}{r} - \frac{2\beta'}{r} + \frac{1}{r^2}\right) e^{-2\beta} = 0,$$

$$-2\left(\dot{\gamma}' - \dot{\gamma}\alpha' - \gamma'\dot{\beta} - \gamma'\dot{\gamma} - \frac{\dot{\beta}}{r}\right) e^{-\alpha-\beta} = 0,$$

$$\frac{1}{r^2} + \left(2\ddot{\gamma} + \dot{\gamma}^2 - 2\dot{\gamma}\dot{\alpha}\right) e^{-2\alpha} - \left(3\gamma'^2 + 2\gamma'\alpha' + \frac{4\gamma'}{r} + \frac{2\alpha'}{r} + \frac{1}{r^2}\right) e^{-2\beta} = 0,$$

$$\left(\ddot{\beta} + 2\ddot{\gamma} + \dot{\beta}^2 + \dot{\gamma}^2 + 2\dot{\gamma}\dot{\beta} - 2\dot{\gamma}\dot{\alpha} - \dot{\alpha}\dot{\beta}\right)e^{-2\alpha}$$
$$- \left(\alpha'' + 2\gamma'' + \alpha'^2 + \gamma'^2 + 2\gamma'\alpha' - 2\gamma'\beta' - \alpha'\beta' + \frac{2\gamma'}{r} + \frac{\alpha'}{r} - \frac{\beta'}{r}\right) e^{-2\beta} = 0.$$

Vacuum Solution

The general vacuum solution is

$$ds^2 = \left(1 - \frac{a}{r\phi}\right) dt^2 - \frac{(r\phi)'^2}{\left(1 - \frac{a}{r\phi}\right)} dr^2 - r^2\phi^2 \left(d\theta^2 + \sin^2\theta \, d\varphi^2\right) . \tag{36}$$

where $\phi = \phi(r)$ is arbitrary. Use this gauge freedom to obtain a Schwarzschild-type solution, i.e. such that $A(r) = 1/B(r)$. This yields

$$\boxed{ds^2 = \phi^2(r) \left[\mu(r) \, dt^2 - \frac{dr^2}{\mu(r)} - r^2 \left(d\theta^2 + \sin^2\theta \, d\varphi^2\right)\right] ,} \tag{37}$$

where

$$\phi(r) = \left(1 - \frac{r}{r_L}\right)^{-1} \quad \text{and} \quad \mu(r) = \left(1 - \frac{r}{r_L}\right)^2 \left(1 - \frac{a}{r} + \frac{a}{r_L}\right) = \left(1 - \frac{r}{r_L}\right)^2 \frac{1 - \frac{r_S}{r}}{1 - \frac{r_S}{r_L}} .$$

Note that the exact Schwarzschild form can be obtained through the transformation

$$\tilde{r} = r\phi(r). \tag{38}$$

but also as the particular case of Eq. (37) in the limit $r_L \to \infty$. Therefore, the parameter r_S may be consistently regarded as the natural generalization of the Schwarzschild radius.

If the line element (37) is interpreted as an exterior solution, the function $\mu(r)$ must remain positive, which holds for $r_S < r$. Additionally, $\bar{r} > 0$ implies $r < r_L$. Cosmological observations indicate the existence of structures on extremely large scales, implying that spacetime is either infinite or bounded by a value of r_L far beyond the currently observed scales. The latter scenario is mathematically plausible, provided that cosmological observations occur in a range $r_S \ll r \ll r_L$—a limit where the coordinates r and $\bar{r}$ are practically indistinguishable. Because of the significant implications, expression (37) is worthy of attention, and it will be referred to as the *Lyra-Schwarzschild* line element.

In order to calculate the effective Newtonian potential, Taylor-expand the functions ϕ and μ in the regime $r_S \ll r \ll r_L$. To second order, this computation produces the equation of motion

$$\frac{d^2 r}{dt^2} = -\frac{m_G}{r^2} + \frac{3m_G}{r_L^2}\left(1 - \frac{r_S}{r_L}\right) - \frac{3r}{r_L^2}, \tag{39}$$

where m_G is the geometric mass:

$$m_G = \frac{\bar{r}_S}{2} = \frac{r_S}{2\left(1 - \frac{r_S}{r_L}\right)} \approx \frac{r_S}{2}\left(1 + \frac{r_S}{r_L}\right), \tag{40}$$

Eq. (39) shows that, in LyST geometry, the geometric mass—acting as the primary effective gravitational source—depends on both r_S and r_L. Moreover, the second-order corrections give rise to a constant repulsive acceleration and a linear term in r, analogous to an anti–de Sitter contribution.

Finally, the equation of motion in the Newtonian limit for a Schwarzschild spacetime with cosmological constant Λ reads

$$\frac{d^2 r}{dt^2} = -\frac{m_G}{r^2} + \frac{\Lambda r}{3}. \tag{41}$$

By comparing Eqs. (39) and (41) when $r_S \ll r$, one finds that the cosmological constant is related to the Lyra radius through

$$\Lambda \approx -\frac{9}{r_L^2}, \tag{42}$$

showing that the spacetime behaves as in anti-de Sitter geometry $r_S \ll r \ll r_L$. These results indicate that LyST geometry naturally reproduces the expected Schwarzschild behavior at short distances while introducing a well-defined large-distance modification that becomes relevant only near the geometric boundary set by r_L.

The Hamilton-Jacobi Approach

The Lagrangian and Hamiltonian of a free particle in LyST theory are given by

$$L(x,\dot{x},\lambda) = \frac{1}{2} m\phi^2 g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu, \quad \text{and} \quad H(x,p,\lambda) = \frac{1}{2m\phi^2} g^{\mu\nu} p_\mu p_\nu, \tag{43}$$

where m is the rest mass, $\dot{x}^\mu = dx^\mu/d\lambda$ are the velocities, and $p_\mu = \partial L/\partial \dot{x}^\mu$ are the canonical conjugate momenta. If λ is an affine parameter,

$$\phi^2 g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu = \varepsilon = \begin{cases} 1, & \text{massive particles;} \\ 0, & \text{photons.} \end{cases} \tag{44}$$

The Hamilton principal function $S(x, P, \lambda)$ generates a canonical transformation from the coordinates (x, p) to the new coordinates (X, P), where X^μ and P_μ are constants. The canonical relations are given by

$$p_\mu = \frac{\partial S(x,P,\lambda)}{\partial x^\mu} \quad \text{and} \quad X^\mu = \frac{\partial S(x,P,\lambda)}{\partial P_\mu}. \tag{45}$$

The evolution of the system is governed by the Hamilton–Jacobi equation:

$$H\left(x, \frac{\partial S}{\partial x}, \lambda\right) + \frac{\partial S}{\partial \lambda} = 0. \tag{46}$$

The general solution of (46) is:

$$S = -\frac{\varepsilon m \lambda}{2} + Et - \ell\varphi \pm \int dr \sqrt{\frac{E^2}{\mu^2} - \frac{\varepsilon m^2 \phi^2}{\mu} - \frac{\ell^2}{\mu r^2}}, \tag{47}$$

Here, the coordinate θ does not appear, as the motion is confined to a plane passing through the origin, which can be chosen such that $\theta = \frac{\pi}{2}$. Now, by replacing S in the second relation of (45), one obtains:

$$t = \pm k \int \frac{dr}{\mu\sqrt{k^2 - \varepsilon\mu\phi^2 - \frac{h^2\mu}{r^2}}}, \tag{48a}$$

$$\lambda = \pm \int \frac{\phi^2\, dr}{\sqrt{k^2 - \varepsilon\mu\phi^2 - \frac{h^2\mu}{r^2}}}, \tag{48b}$$

$$\varphi = \pm h \int \frac{dr}{r^2\sqrt{k^2 - \varepsilon\mu\phi^2 - \frac{h^2\mu}{r^2}}}, \tag{48c}$$

where $k = E/m$ and $h = \ell/m$ are the specific energy and the specific angular momentum.

Massive Particles

For massive particles, $\varepsilon = 1$ and $\lambda = \tau$ is the proper time. From (48a):

$$\mathcal{E} = \frac{1}{2}\dot{\tilde{r}}^2 + V_{\text{eff}} = \frac{1}{2}\phi^4 \dot{r}^2 + V_{\text{eff}}, \tag{49}$$

where $\mathcal{E} = (k^2 - 1)/2$. Here, the quantity V_{eff} is the effective potential

$$V_{\text{eff}} := -\frac{m_G}{\tilde{r}} + \frac{h^2}{2\tilde{r}^2} - \frac{m_G h^2}{\tilde{r}^3} \tag{50}$$

$$= -\frac{m_G}{r}\left(1 - \frac{r}{r_L}\right) + \frac{h^2}{2r^2}\left(1 - \frac{r}{r_L}\right)^2 - \frac{m_G h^2}{r^3}\left(1 - \frac{r}{r_L}\right)^3 . \tag{51}$$

Moreover, from (48c), one obtains:

$$\frac{d^2\tilde{u}}{d\varphi^2} = -\frac{1}{h^2}\frac{dV_{\text{eff}}}{d\tilde{u}} = \frac{m_G}{h^2} - \tilde{u} + 3m_G\tilde{u}^2 , \tag{52}$$

whose solution for $\tilde{u} = 1/\tilde{r}$ is the standard Schwarzschild result.
Furthermore, the solution for $u = 1/r$ follows from $u = \tilde{u} + u_L$, with $u_L = 1/r_L$.

The different types of motion can be determined by analyzing the behavior of the effective potential as a function of the parameters ℓ and m_G. To this end, it is necessary to examine the first and second derivatives of the effective potential:

$$\frac{dV_{\text{eff}}}{dr} = \frac{d\tilde{r}}{dr}\frac{dV_{\text{eff}}}{d\tilde{r}} = \phi^2 \frac{dV_{\text{eff}}}{d\tilde{r}}, \tag{53}$$

$$\frac{d^2 V_{\text{eff}}}{dr^2} = \frac{d^2\tilde{r}}{dr^2}\frac{dV_{\text{eff}}}{d\tilde{r}} + \left(\frac{d\tilde{r}}{dr}\right)^2 \frac{d^2 V_{\text{eff}}}{d\tilde{r}^2} = \frac{2\phi^3}{r_L}\frac{dV_{\text{eff}}}{d\tilde{r}} + \phi^4 \frac{d^2 V_{\text{eff}}}{d\tilde{r}^2}. \tag{54}$$

Since $\phi \neq 0$, the number of critical points is the same in both coordinate systems,

$$\left(\frac{dV_{\text{eff}}}{dr}\right)_{r_{crit}} = \left(\frac{dV_{\text{eff}}}{d\tilde{r}}\right)_{\tilde{r}_{crit}} = 0, \tag{55}$$

as well as their stability,

$$\operatorname{sgn}\left(\frac{d^2 V_{\text{eff}}}{dr^2}\right)_{r_{crit}} = \operatorname{sgn}\left(\frac{d^2 V_{\text{eff}}}{d\tilde{r}^2}\right)_{\tilde{r}_{crit}}. \tag{56}$$

From Eq. (55), the critical points are given by

$$r_{\pm} = 6m_G \left[\left(1 + \frac{6m_G}{r_L}\right) \mp \sqrt{1 - \frac{12m_G^2}{h^2}} \right]^{-1}. \tag{57}$$

- $h^2 < 12m_G^2$: No critical points.
- $h^2 = 12m_G^2$: One inflection point at $r_{inf} \equiv 6m_G \left(1 + \frac{6m_G}{r_L}\right)^{-1}$.
- $h^2 > 12m_G^2$: Two equilibrium points: r_+ (stable) and r_- (unstable).

On the other hand, the roots of the effective potential are given by

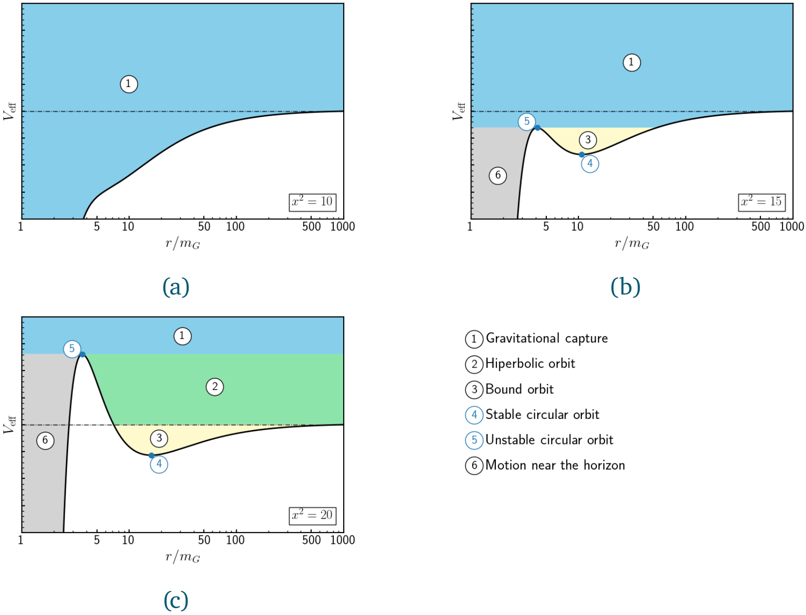
$$R_{\pm} = 4m_G \left[\left(1 + \frac{4m_G}{r_L}\right) \mp \sqrt{1 - \frac{16m_G^2}{h^2}} \right]^{-1}. \tag{58}$$

- $h^2 < 16m_G^2$: No real roots.
- $h^2 = 16m_G^2$: One root, given by $R_0 \equiv 4m_G \left(1 + \frac{4m_G}{r_L}\right)^{-1}$.
- $h^2 > 16m_G^2$: Two different roots.

In addition, the following asymptotic behavior is observed:

$$V_{\text{eff}}(\tilde{r} \to \infty) \equiv V_{\text{eff}}(r \to r_L) \to 0^-, \tag{59}$$

$$V_{\text{eff}}(\tilde{r} \to 0) \equiv V_{\text{eff}}(r \to 0) \to -\infty. \tag{60}$$



$V_{\rm eff}$
r/m_G
1
5
10
50
100
500
1000
$x^2 = 10$
$x^2 = 15$
$x^2 = 20$
(a)
(b)
(c)
1 Gravitational capture
2 Hiperbolic orbit
3 Bound orbit
4 Stable circular orbit
5 Unstable circular orbit
6 Motion near the horizon

Figure 1:

Photons

In the case of photons, $\varepsilon = 0$, and we cannot define proper time. From (48b):

$$\frac{m_G^2}{b^2} = \frac{m_G^2}{h^2}\phi^4 \dot{r}^2 + V_{\rm eff}, \tag{61}$$

where $b = h/k$. Here, the effective potential is identified as:

$$V_{\rm eff} := \frac{m_G^2}{\tilde{r}^2}\left(1 - \frac{2m_G}{\tilde{r}}\right) \tag{62}$$

$$= \frac{m_G^2}{r^2}\left(1 - \frac{r}{r_L}\right)^2 \left(1 - \frac{2m_G}{r} + \frac{2m_G}{r_L}\right). \tag{63}$$

Once again, from (48c), one obtains:

$$\frac{d^2\tilde{u}}{d\varphi^2} = -\frac{1}{2m_G^2}\frac{dV_{\rm eff}}{d\tilde{u}} = -\tilde{u} + 3m_G\tilde{u}^2, \tag{64}$$

whose solution for $\tilde{u} = 1/\tilde{r}$ is the standard Schwarzschild result and $u = \tilde{u} + u_L$.

In this case, there is only one critical point:

$$\tilde{r}_c = 3m_G \qquad \text{or} \qquad r_c = \frac{3m_G}{1 + \frac{3m_G}{r_L}}, \tag{65}$$

which is an unstable equilibrium point, giving the maximum value for the potential:

$$V_{\text{eff}}^{max} = V_{\text{eff}}(r_c) = \frac{1}{27}. \tag{66}$$

On the other hand, there is also only one root of the effective potential, which is precisely the Schwarzschild radius:

$$\tilde{r}_S = 2m_G \qquad \text{or} \qquad r_S = \frac{2m_G}{1 + \frac{2m_G}{r_L}}. \tag{67}$$

Moreover, the following asymptotic behavior is observed:

$$V_{\text{eff}}(\tilde{r} \to \infty) \equiv V_{\text{eff}}(r \to r_L) \to 0^+ \,, \tag{68}$$

$$V_{\text{eff}}(\tilde{r} \to 0) \equiv V_{\text{eff}}(r \to 0) \to -\infty \,. \tag{69}$$

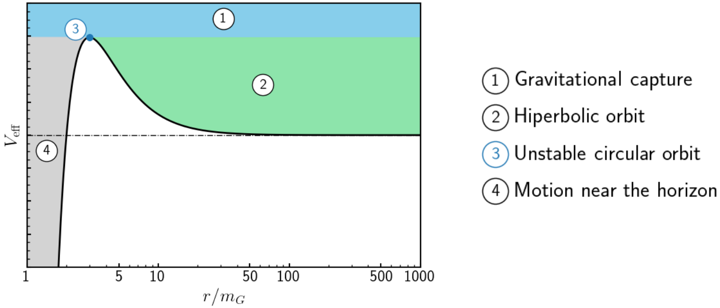

1
3
2
4
V_{eff}
1
5
10
50
100
500
1000
r/m_G
1 Gravitational capture
2 Hiperbolic orbit
3 Unstable circular orbit
4 Motion near the horizon

Figure 2:

Consider light source and an observer, both with fixed spatial coordinates $(r_E, \theta_E, \varphi_E)$ and $(r_R, \theta_R, \varphi_R)$, respectively. The source emits a pulse at time t_E and received by the observer at time t_R. Another pulse is emitted by the source at time $(t_E + \Delta t_E)$ and received by the observer at time $(t_R + \Delta t_R)$. The photons will follow null geodesics and, by choosing a parameterization λ, one can write:

$$t_R - t_E = \int_{\lambda_E}^{\lambda_R} d\lambda\, \mu^{-1/2} \left(-g_{ij} \frac{dx^i}{d\lambda} \frac{dx^j}{d\lambda} \right)^{1/2} . \tag{70}$$

This integral depends only on the initial and final points, so it is the same for both pulses:

$$t_R - t_E = (t_R + \Delta t_R) - (t_E + \Delta t_E) \qquad \Rightarrow \qquad \Delta t_E = \Delta t_R . \tag{71}$$

Now, both the source and the observer are fixed in space. So the proper time of their curves is $d\tau^2 = ds^2 = \phi^2 g_{00}\, dt^2$. Thus,

$$\frac{\Delta \tau_E}{\Delta \tau_R} = \frac{\sqrt{\phi^2(r_E)\mu(r_E)}\Delta t_E}{\sqrt{\phi^2(r_R)\mu(r_R)}\Delta t_R} = \sqrt{\frac{1 - \frac{2m_G}{r_E} + \frac{2m_G}{r_L}}{1 - \frac{2m_G}{r_R} + \frac{2m_G}{r_L}}} . \tag{72}$$

The gravitational redshift is defined in terms of the emission and reception wavelengths:

$$z = \frac{\lambda_R - \lambda_E}{\lambda_E}. \tag{73}$$

Here, λ represents a wavelength instead of the parameter in Eq. (70). Wavelengths and frequencies are related through $\lambda \propto \nu^{-1} \propto \Delta\tau$. Therefore, the general form of the gravitational redshift in Lyra spacetime is:

$$z = \sqrt{\frac{1 - \frac{2m_G}{r_R} + \frac{2m_G}{r_L}}{1 - \frac{2m_G}{r_E} + \frac{2m_G}{r_L}}} - 1. \tag{74}$$

In particular, when the light source and the observer are far away from the gravitational source, such that $r_E, r_R \gg r_S$, expression (74) simplifies to

$$z \approx \frac{m_G}{r_E}\left(1 - \frac{r_E}{r_R}\right). \tag{75}$$

Redshift, properly speaking, occurs when $r_E > r_R$, while blueshift occurs in the opposite case. This remains true for the general expression (74).

The radial motion ($h = 0$) for photons is determined by Eq. (48a), whose solution is given by

$$t_{\pm} = \tau \pm \left[\frac{r}{1 - \frac{r}{r_L}} + 2m_G \ln \left(\frac{r}{1 - \frac{r}{r_L}} - 2m_G \right) \right], \tag{76}$$

where τ is an integration constant. Note that this expression diverges for $r = r_S$ and $r = r_L$. However, the only physical singularity is $r = 0$, as one can verify from the expression of the Kretschmann scalar:

$$K = \frac{48 m_G^2}{\tilde{r}^6} = \frac{12 r_S^2}{r^6} \frac{\left(1 - \frac{r}{r_L}\right)^6}{\left(1 - \frac{r_S}{r_L}\right)^2}. \tag{77}$$

Therefore, $r = r_S$ and $r = r_L$ are just coordinate singularities, as shown in Figure 3.

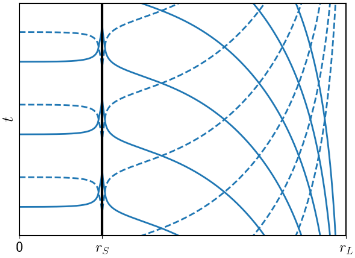


Figure 3:

In this figure continuous lines represent ingoing photons and dashed lines represent outgoing photons. The singularity $r = r_S$ can be removed in a similar way to the Schwarzschild case. Meanwhile, the singularity $r = r_L$ is a consequence of the transformation $\tilde{r} = r\phi(r)$. This makes sense when considering that $r = r_L$ corresponds to $\tilde{r} = \infty$, which is neither a physical singularity nor a physical place.

Final Remarks: Main Results

Lyra Scalar-Tensor gravity (LyST) is formulated on a Lyra manifold with a torsion-free, metric-compatible connection. The fundamental dynamical fields are the metric $g_{\mu\nu}$ and the scale function ϕ, making LyST a natural extension of General Relativity with an additional geometrical degree of freedom.

The theory:

- Proves Birkhoff theorem under certain considerations on the scale function.
- Recovers Newtonian gravity in the weak-field and static limit.
- Subleading corrections introduce repulsive and anti–de Sitter–like effects, potentially relevant at very large scales.
- Some astronomical uncertainties allow lower bounds of the order $r_L \gtrsim 10^{21}\,\mathrm{m}$.

The most general spherically symmetric solution depends on a free scale function, but it is physically equivalent to the Schwarzschild solution via the coordinate transformation $\tilde{r} = r\,\phi(r)$.

Thanks For Your Attention!