Memory Kernel and CP-Divisibility of Gaussian Collisional Models[1]

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Collisional Model

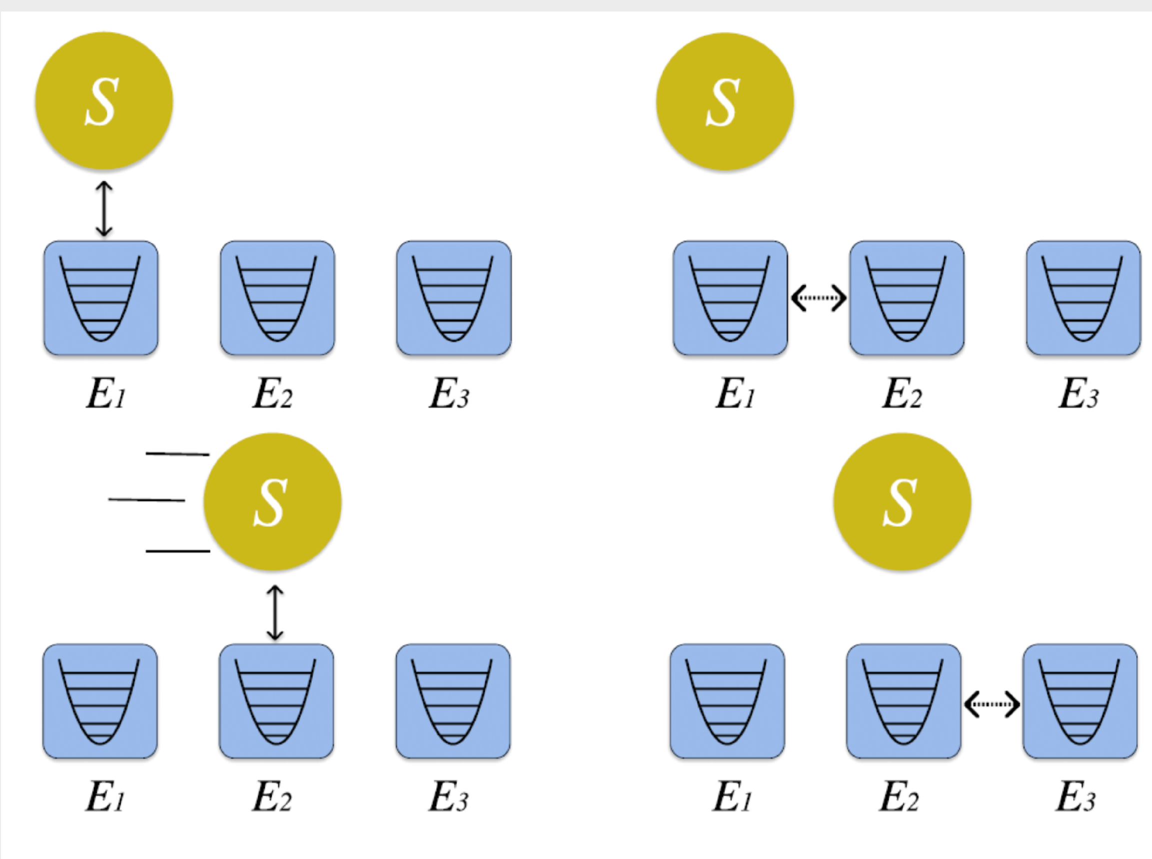

FIG. 1. First steps of the studied non-Markovian collisional model.

We consider the figure's initially uncorrelated collisional model where the system-ancilla interactions SE_n are interpersed by ancilla-ancilla interactions E_nE_{n+1} under two gaussian Hamiltonians: beam splitter BS and two-mode squeezing TMS. Setting SE_n as BS interaction while allowing E_nE_{n+1} to be either, we study the memory effects generated by the dynamics. As the global evolution of SE_1E_2... is unitary, we can translated into sympletic form for the total covariance matrix σ:

$$\sigma^n = S_{n,n+1} S_n \sigma^{n-1} S_n^{\mathsf{T}} S_{n,n+1}^{\mathsf{T}}$$

where S_n and $S_{n,n+1}$ are the sympletic matrices associated with the unitaries SE_n and E_nE_{n+1} correspondingly.

Markovian Embedding:

The resulting dynamics at any time involves only SE_nE_{n+1}. Ancillas with $m \geq n+2$ did not participate yet and the ancillas with $m < n$ will never participate again. Therefore, we define the reduced CM of SE_{n+1}

$$\gamma^n = \begin{pmatrix} \theta^n & \xi^n_{n+1} \\ \xi^{n,\mathsf{T}}_{n+1} & \epsilon^n_{n+1} \end{pmatrix},$$

where θ^n is the system's CM; ϵ^n, the ancilla's; and ξ^n, the correlation between them. Eq. (1) can be cast as

$$\gamma^{n+1} = X\gamma^n X^{\mathsf{T}} + Y,$$

where X, Y are 4 by 4 matrices referring to the unitaries. We plot the BS and TMS maps where λ_s refers to the SE_n interaction strength, and λ_e, ν_e to the E_nE_{n+1}.

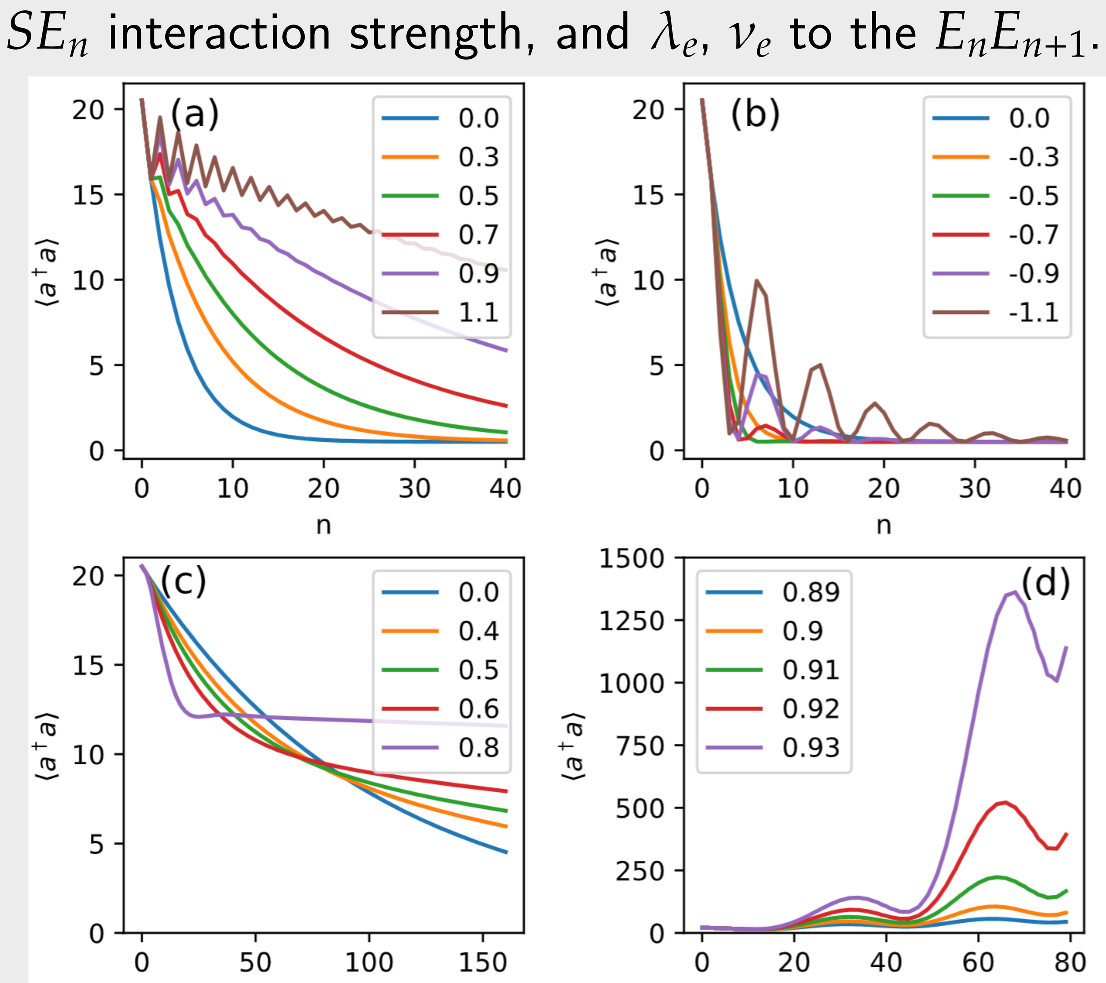

FIG. 2. Number of excitations in the system as a function of time. (a,b) BS dynamics with $\lambda_s = 0.5$ and different values of λ_e (with $\lambda_e > 0$ in (a) and $\lambda_e < 0$ in (b)). (c,d) Same, but for the TMS dynamics, with $\lambda_s = 0.1$ and different values of ν_e (with $\nu_e < \nu_e^{\text{crit}}$ in (a) $\nu_e \geqslant \nu_e^{\text{crit}}$ in (b), where $\nu_e^{\text{crit}} = \sinh^{-1}(1) \simeq 0.8813$). The ancillas start in the vacuum, and the system in a thermal state with $\langle a^\dagger a\rangle^0 = 20$.

Memory Kernel

A much older notion of non-Markovianity is that of a memory kernel $\mathcal{K}_{t-t'}$, as present already in the seminal works of Nakajima and Zwanzig

$$\frac{d\rho_S}{dt} = -i[H_S, \rho_S] + \int_0^t \mathcal{K}_{t-t'}[\rho(t')]\, dt'$$

Within our framework, one may equivalently formulate a memory kernel acting only on the system's covariance matrix θ. This can be accomplished by writing down the difference equation for the system's only:

$$\theta^{n+1} = x^2\theta^n + \sum_{r=0}^{n-1} \mathcal{K}_{n-r-1}(\theta^r) + G_n$$

We can write more explicitly in terms of a Kraus operator-sum representation:

$$\mathcal{K}_n(\theta) = \sum_{ij} k^n_{ij} M_i\, \theta\, M_j^{\mathsf{T}}$$

where k^n_{ij} are coefficients that depend on time and $\{M_i\}$ are a complete 2 by 2 set of matrices $\{\mathbb{I}_2, \sigma_z, \sigma_+, \sigma_-\}$.

The memory itself is contained in the dependence of k^n_{ij} on n. The subscripts i, j determines how different elements of θ^r affect θ^n. For instance, in the BS map, the only non-zero coefficient will be the one proportional to $\mathbb{I}_2\theta\mathbb{I}_2$ which we refer to as k^n_{11}, i.e. $\mathcal{K}_n(\theta) = k^n_{11}\theta$.

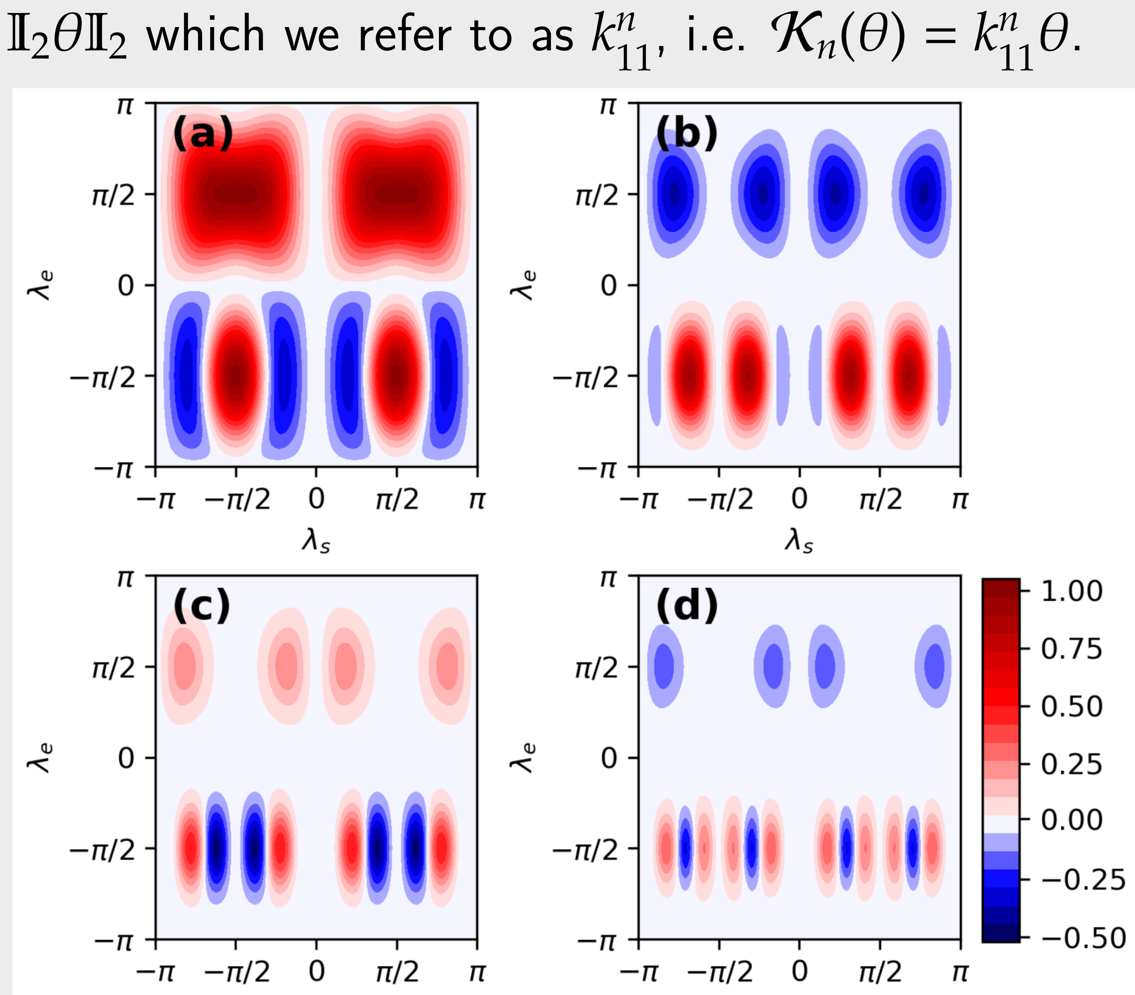

FIG. 3. Diagrams for the memory kernel coefficient k^n_{11} in the (λ_s, λ_e) plane for different values of n, from $n = 0$ to $n = 4$

Converserly, in the TMS map there will be four non-zero coefficients $k^n_{11}, k^n_{1,z}, k^n_{z,1}$ and $k^n_{z,z}$, corresponding to $\mathbb{I}_2$ and σ_z. We plot the memory kernels of Q^2 and P^2 $k^n_q = k^n_{11} + k^n_{1,z} + k^n_{z,1} + k^n_{z,z}$ and $k^n_p = k^n_{11} - k^n_{1,z} - k^n_{z,1} + k^n_{z,z}$

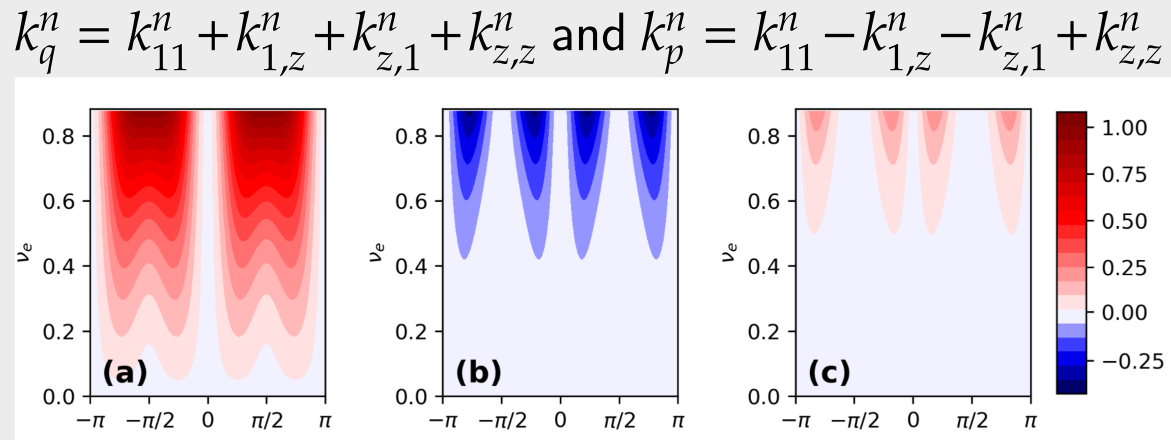

FIG. 4. Diagrams for the memory kernel coefficient k_q of the TMS dynamics in the (λ_s, ν_e) plane for $n = 0$ to $n = 2$

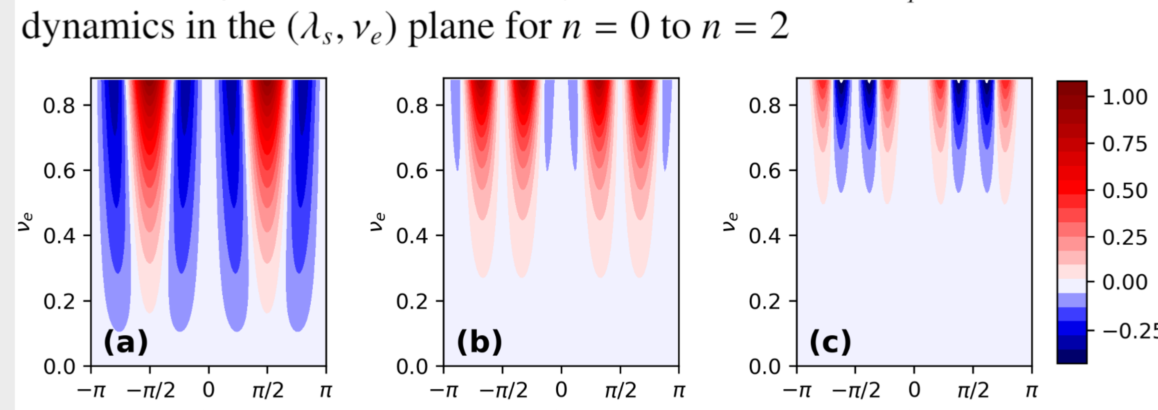

FIG. 5. Diagrams for the memory kernel coefficient k_p of the TMS dynamics in the (λ_s, ν_e) plane for $n = 0$ to $n = 2$

CP-Divisibility

Map divisibility is the ultimate test of non-Markovianity. For Gaussian dynamics, any CPTP map must have the form $\theta \to \mathcal{X}\theta\mathcal{X}^{\mathsf{T}} + \mathcal{Y}$ where the matrices satisfies

$$\mathcal{M}[\mathcal{X}, \mathcal{Y}] := 2\mathcal{Y} + i\Omega - i\mathcal{X}\Omega\mathcal{X}^{\mathsf{T}} \geq 0$$

with $\Omega = i\sigma_y$ the sympletic form.

In our case, the evolution of the system from initial time 0 to step n has the form

$$\theta^n = \mathcal{X}_n\theta^0\mathcal{X}_n^{\mathsf{T}} + \mathcal{Y}_n$$

The matrices $\mathcal{X}_n$ and $\mathcal{Y}_n$ can be read from the (1,1) block of the general solution

$$\mathcal{X}_n = (X^n)_{11},$$

$$\mathcal{Y}_n = (X^n)_{12}\epsilon(X^n)^{\mathsf{T}}_{12} + \sum_{r=0}^{n-1}\left[X^{n-r-1}Y(X^{\mathsf{T}})^{n-r-1}\right]_{11}$$

To probe whether the dynamics is divisible, we consider the mapping taking the system from n to m $(m > n)$

$$\theta^m = \mathcal{X}_{mn}\theta^n\mathcal{X}_{mn}^{\mathsf{T}} + \mathcal{Y}_{mn}$$

where

$$\mathcal{X}_{mn} = \mathcal{X}_m\mathcal{X}_n^{-1}, \qquad \mathcal{Y}_{mn} = \mathcal{Y}_m - \mathcal{X}_{mn}\mathcal{Y}_n\mathcal{X}_{mn}^{\mathsf{T}}$$

The dynamics is then considered divisible when the intermediate maps are CPTP, that is $\mathcal{M}[\mathcal{X}_{mn}, \mathcal{Y}_{mn}] \geq 0$. The above criteria can be also used as a figure of merit

$$\mathcal{N}_{mn} = \sum_k \frac{|m_k| - m_k}{2}$$

where $m_k = \text{eigs}\big(\mathcal{M}[\mathcal{X}_{mn}, \mathcal{Y}_{mn}]\big)$

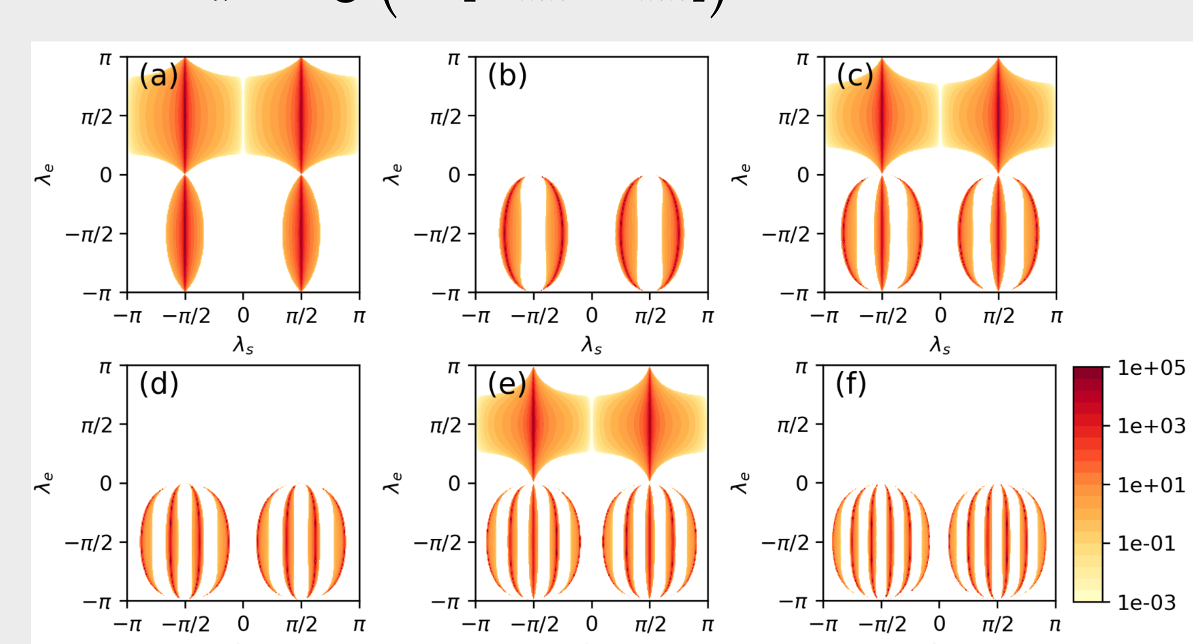

FIG. 6. CP-Divisibility measure $\mathcal{N}_{n+1,n}$ in the (λ_s, λ_e) plane for the BS dynamics. Each plot corresponds to a different value of n from 1 to 6 in steps of 1.

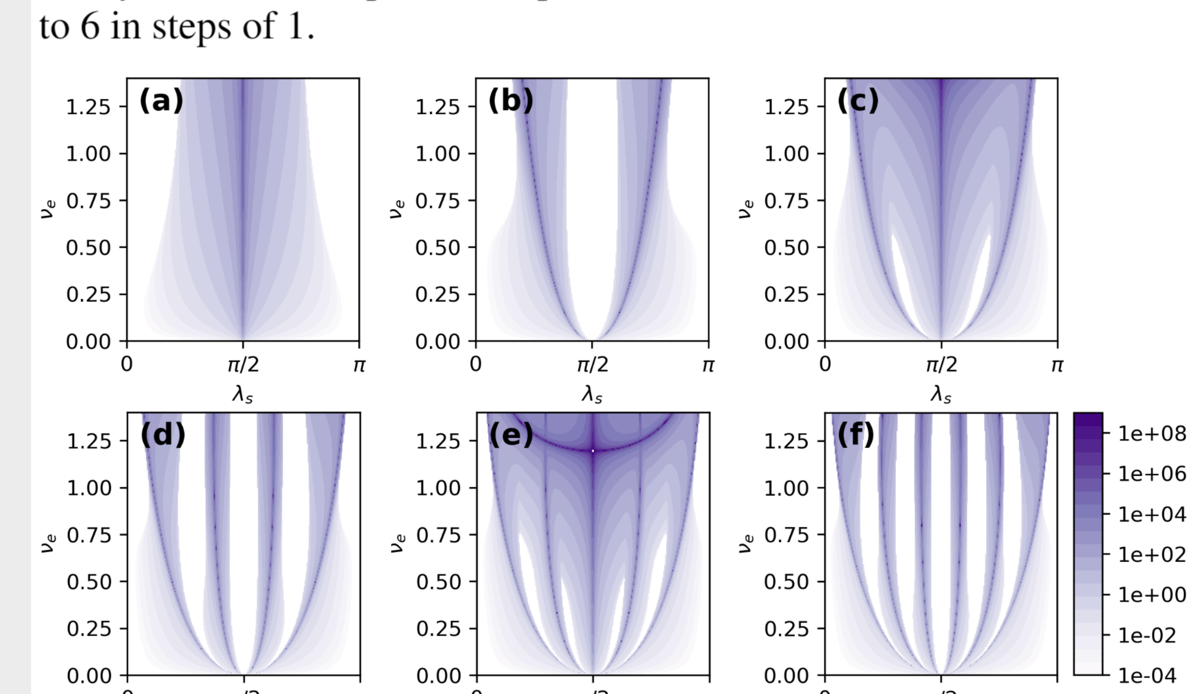

FIG. 7. CP-Divisibility measure $\mathcal{N}_{n+1,n}$ in the (λ_s, ν_e) plane for the TMS dynamics. Each plot corresponds to a different value of n from 1 to 6 in steps of 1.

Acknowledgments:

FAPESP CNPq

The authors acknowledge financial support from Brazilian agency CNPq (grant 157168/2018-2) and the São Paulo Research Foundation (grants 2017/07973-5, 2017/50304-7, and 2018/12813-0).

References

[1] R. R. Camasca and G. T. Landi, Memory kernel and divisibility of gaussian collisional models, arXiv preprint arXiv:2008.00765 (2020).