



## The Lane-Emden Equation

The Lane-Emden equation:

$$\frac{1}{\xi^2} \frac{d}{d\xi} \left( \xi^2 \frac{d\theta}{d\xi} \right) = -\theta^n \quad (1)$$

is of interest in physics due to its applications in various fields as astrophysics, quantum mechanics and kinetic theory.

In astrophysics, the Lane-Emden equation provides us with a detailed explanation of the astrophysical properties of these stars based on newtonian self-gravitating, spherically symmetric and polytropic fluid [1]. To derive it, we begin with the equations of mass continuity and of hydrostatic equilibrium. Since there are three unknowns (pressure, density, and mass as a function of radius) and only two equations, we need to introduce an additional equation. For polytropes such equation is provided by the pressure-density relation

$$P = K\rho^{1+\frac{1}{n}}. \quad (2)$$

This adds a third equation, and the set of three equations can then be reduced to a single differential equation whose terms depend on  $n$  and on  $K$ , which after an scale transform gives us ec. 1

## Sympy

Sympy is a Python library for symbolic mathematics. It aims to become a full-featured computer algebra system (CAS) while keeping the code as simple as possible in order to be comprehensible and easily extensible [2]. Although Sympy is not a necessary tool to solve the Lane-Emden equation, in this work we use it to show the capacity of the tool as a support in solving other problems that may require the use of a CAS due to the density of the algebraic manipulations required to address them.

With Sympy we can define the Lane-Emden equation as follows.

```
lhs = simplify(
    (1 / xi ** 2) * Derivative(
        (xi ** 2) * Derivative(
            theta(xi), xi
        ), xi
    ).doit()
)
rhs = -theta(xi) ** n
lane_emden_eq = Equation(lhs, rhs)

$$\frac{d^2}{d\xi^2}\theta(\xi) + \frac{2}{\xi}\frac{d}{d\xi}\theta(\xi) = -\theta^5(\xi)$$

```

## Solutions for $n = 0$

```
lane_emden_eq_0 = lane_emden_eq.subs(n, 0)
solution = dsolve(lane_emden_eq_0, theta(xi))
solution = solution.subs(solve(
    [
        simplify(xi * solution.rhs).subs(xi, 0),
        Derivative(
            simplify(xi * solution.rhs), xi
        ).doit().subs(xi, 0) - 1,
    ],
    symbols('C1_C2'),
)).simplify()

$$\theta(\xi) = 1 - \frac{\xi^2}{6}$$

```

## Solutions for $n = 1$

```
lane_emden_eq_0 = lane_emden_eq.subs(n, 0)
solution = dsolve(lane_emden_eq_0, theta(xi))
solution = solution.subs(solve(
    [
        simplify(xi * solution.rhs).subs(xi, 0),
        Derivative(
            simplify(xi * solution.rhs), xi
        ).doit().subs(xi, 0) - 1,
    ],
    symbols('C1_C2'),
)).simplify()

$$\theta(\xi) = \frac{\sin(\xi)}{\xi}$$

```

## Solutions for $n = 5$

Although for  $n = 0$  and  $n = 1$ , sympy solutions are straightforward, for  $n = 5$  the process is longer. For such reason here we only present the real solutions we obtained. To proceed we need to transform ec. 1 to its autonomous form (ec. 3) which depends on a parameter  $C$  [3].

$$\left(\frac{dz}{dt}\right) = \frac{1}{12}(-z^6 + 3z^2 + C) \quad (3)$$

►  $C = -2$

$$\theta(\xi) = \pm (\sqrt{2\xi})^{-1}$$

►  $-2 < C < 0$

$$\theta(\xi) = \pm \sqrt{\frac{aby^2}{2\xi(b(y^2-1)+a)}}, y = \text{dc}\left(\frac{1}{2}\sqrt{\frac{(a+c)b}{3}}\ln(B\xi), \sqrt{\frac{(b-a)c}{(a+c)b}}\right)$$

►  $C = 0$

$$\theta(\xi) = \pm (\sqrt{1+\xi^2/3})^{-1}$$

►  $0 < C < 2$

$$\theta(\xi) = \pm \sqrt{\frac{acy^2}{2\xi(a(y^2+1)+c)}}, y = \text{dc}\left(\frac{1}{2}\sqrt{\frac{(a+c)b}{3}}\ln(B\xi), \sqrt{\frac{(b-a)c}{(a+c)b}}\right)$$

►  $C = 2$

$$\theta(\xi) = \pm \sin(\ln \sqrt{\xi}) \left(\sqrt{3\xi + 2\xi \sin^2(\ln \sqrt{\xi})}\right)^{-1}$$

►  $2 < C$

$$\theta(\xi) = \pm \sqrt{C} \left(\sqrt{2\xi} \sqrt{\rho(\ln(B\xi)/(2\sqrt{3}); 12, 4(C^2-2)) - 1}\right)^{-1}$$

Where both dc and sc are subsidiary Jacobian elliptic functions,  $\rho$  is the Weierstrass elliptic function,  $B$  is an integration constant and

$$a = 2 \sin\left(\frac{1}{3} \arcsin\left(\frac{|C|}{2}\right)\right), b = 2 \cos\left(\frac{1}{3} \arccos\left(-\frac{|C|}{2}\right)\right),$$

$$c = 2 \cos\left(\frac{1}{3} \arccos\left(\frac{|C|}{2}\right)\right),$$

For  $C < -2$  there are no real solutions.

## References

- [1] S. Chandrasekhar. *An Introduction to the Study of Stellar Structure*. Dover Publications, 2010.
- [2] Aaron Meurer, Christopher P. Smith, Mateusz Paprocki, Ondřej Čertík, Sergey B. Kirpichev, Matthew Rocklin, AMIT Kumar, Sergiu Ivanov, Jason K. Moore, Sartaj Singh, Thilina Rathnayake, Sean Vig, Brian E. Granger, Richard P. Muller, Francesco Bonazzi, Harsh Gupta, Shivam Vats, Fredrik Johansson, Fabian Pedregosa, Matthew J. Curry, Andy R. Terrel, Štěpán Roučka, Ashutosh Saboo, Isuru Fernando, Sumith Kulal, Robert Cimrman, and Anthony Scopatz. Sympy: symbolic computing in python. *PeerJ Computer Science*, 3:e103, January 2017.
- [3] Patryk Mach. All solutions of the  $n = 5$  laneemden equation. *Journal of Mathematical Physics*, 53(6):062503, Jun 2012.